Applications of the general Lyapunov ISS small-gain theorem for networks

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***Abstract*— We recall the definitions of input-to-state-stability Lyapunov functions and general small gain theorems. These are then exemplarily used to prove input-to-state stability of and to construct ISS Lyapunov functions for four areas of applications: Linear systems, a Cohen-Grossberg neuronal network, error dynamics in formation control, as well as nonlinear transistor-linear resistor circuits.**

Keywords: Input-to-state stability (ISS), Lyapunov functions, general small-gain theorem, constructive method, example applications

I. INTRODUCTION

For nonlinear systems it is often difficult to prove stability properties. Knowledge of a Lyapunov function makes that task simple, but finding a Lyapunov function is an art of its own and generally also a very complicated task, especially for high dimensional systems.

One approach is to split complicated high dimensional systems into smaller, interconnected subsystems evolving in lower dimensional spaces. The idea is to find Lyapunov functions for the lower dimensional systems and then put these together to obtain a Lyapunov function for the high dimensional system. Thereby taking into account that the subsystems themselves are in a nonlinear fashion interconnected and that there may also exist external inputs or disturbances, that have to be accounted for.

The input-to-state stability (ISS) framework introduced by Sontag in 1989 [13] serves as a general framework stability with respects to inputs. Its equivalent Lyapunov characterization [14] and the corresponding small gain theorem [9] or its generalization [6], [8] can be implemented to establish exactly the idea detailed above: A bottom-to-top stability analysis of high dimensional nonlinear systems.

We consider several example applications of general ISS small gain theorems provided by the authors in [6], [7], [5], [3]. In particular we apply the nonlinear Lyapunov ISS small-gain theorem for networks to a Cohen-Grossberg neuronal network, and error dynamics in formation control.

The paper is organized as follows. In the next section we recall the necessary definitions and notation. In Section III we quote the small gain type theorems related to ISS networks. The main results of this paper are four example applications that are given in Section IV. Section V concludes the paper.

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II. NOTATION AND DEFINITIONS

Consider several nonlinear systems with inputs given by

$$\Sigma_i : \dot{x}_i = f_i(x_1, \ldots, x_n, u_i), \quad i = 1, \ldots, n, \tag{1}$$

where $x_i \in \mathbb{R}^{N_i}$, $u_i \in \mathbb{R}^{M_i}$. The functions $f_i : \mathbb{R}^{\sum_j N_j + M_i} \to \mathbb{R}^{N_i}$, $i = 1, \ldots, n$, are assumed to be such that there exists a unique solution for any given initial condition and any essentially bounded, measurable inputs $x_j, j \neq i$, u_i. We say that x_j is an internal input for the system Σ_i with $i \neq j$ and u_i is an external input to Σ_i. Denoting $x = (x_1^T, \ldots, x_n^T)^T \in \mathbb{R}^N$, $N = \sum_{i=1}^n N_i$, $u = (u_1^T, \ldots, u_n^T)^T$, $f(x,u) = (f_1(x,u_1)^T, \ldots, f_n(x,u_n)^T)^T$ we can represent the interconnection in the form

$$\dot{x} = f(x, u). \tag{2}$$

An appropriate stability notion for such systems is input-to-state stability as introduced in [13] and is now commonly used by many authors. To recall the definition we will need the following notation. Let $\mathbb{R}_+ := [0, \infty)$; a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is non-increasing and tends to zero at infinity. Let $|x|$ denote the Euclidean norm of $x \in \mathbb{R}^n$ and $||\cdot||_\infty$ the standard norm in L_∞. By $\mathbb{R}^n_+$ we denote the positive orthant in $\mathbb{R}^n$. For two vectors a and b in $\mathbb{R}^n$ we denote $a \geq b \quad \Leftrightarrow \quad a_i \geq b_i,\ i = 1, \ldots, n$. The relation $a > b$ is defined in the same way. The negation of the relation $a \geq b$ is denoted by $a \not\geq b$. This is equivalent to the statement that there exists at least one $i \in \{1, \ldots, n\}$ such that $a_i < b_i$.

Definition 2.1: System (2) is called input-to-state stable from u to x, if there exists a $\gamma \in \mathcal{K}_\infty$, and a $\beta \in \mathcal{KL}$, such that

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(||u||_\infty) \quad \forall\, t \geq 0, \tag{3}$$

for all initial states $x(0) \in \mathbb{R}^N$ and inputs $u \in L_\infty$. In this case γ is called nonlinear gain.

It is known that ISS defined in this way is equivalent to the existence of an ISS-Lyapunov function defined by

Definition 2.2: A smooth function $V : \mathbb{R}^N \to \mathbb{R}_+$ is called ISS-Lyapunov function of (2) if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}_\infty$, and a positive definite function α such that

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^N, \tag{4}$$

$$V(x) \geq \chi(|u|) \Longrightarrow \nabla V(x) \cdot f(x, u) \leq -\alpha(V(x)). \tag{5}$$

The function χ is then called Lyapunov gain.

There are many nonlinear gains in case of an interconnected system (1) related to different inputs in each subsystem. Typically one says that Σ_i is ISS if any its solution satisfies for some $\gamma_{ui}, \gamma_{ij} \in \mathcal{K}_\infty$ and $\beta_i \in \mathcal{KL}$

$$|x_i(t)| < \beta_i(|x_i(0)|, t) + \sum_{j \neq i} \gamma_{ij}(||x_j||_\infty) + \gamma_{ui}(||u_i||_\infty), \quad t \geq 0$$

or one uses $\max_{j \neq i}$ instead of $\sum_{j \neq i}$. However, it becomes restrictive for some nonlinear systems as we will see in the examples below. We write this definition in a more general way. For this purpose we introduce:

Definition 2.3: A continuous function $\mu : \mathbb{R}^n_+ \to \mathbb{R}_+$ is called a monotone aggregation function if it satisfies

(i) $\mu(0) = 0$.
(ii) $\mu(s) \geq 0$ for all $s \in \mathbb{R}^n_+$ and $\mu(s) > 0$ if $s > 0$.
(iii) for any $s_1 > s_2 \in \mathbb{R}^n_+$ it holds that $\mu(s_1) > \mu(s_2)$.

The space of monotone aggregation functions is denoted by MAF_n and MAF^m_n denotes the set of continuous functions $\mu : \mathbb{R}^n_+ \to \mathbb{R}^m_+$ such that each component function is in MAF_n.

We also define $\gamma_{ii} \equiv 0$ for $i = 1, \dots, n$. Using this notation we say that Σ_i is ISS if its solutions satisfy

$$|x_i(t)| < \mu_i(\beta_i(|x_i(0)|, t), \gamma_{i1}(||x_1||_\infty), \dots, \gamma_{in}(||x_n||_\infty), \gamma_i(||u||_\infty)).$$

Similarly we define an ISS-Lyapunov function for Σ_i:

Assume that for each of the subsystems $\Sigma_j, j = 1, \dots, n$ we are given a proper, positive definite function $V_j : \mathbb{R}^{N_j} \to \mathbb{R}_+$. The function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ is an ISS-Lyapunov function for Σ_i if there exist $\mu_i \in \mathrm{MAF}_n, \gamma_{ij} \in \mathcal{K} \cup \{0\}, \gamma_i \in \mathcal{K}$ and a positive definite function α_i such that

$$V_i(x_i) \geq \mu_i(\gamma_{i1}(V_1(x_1)), \dots, \gamma_{in}(V_n(x_n)), \gamma_i(||u||_\infty)) \implies \nabla V_i(x_i) f_i(x, u) < -\alpha(|x_i|) \tag{6}$$

The gains γ_{ij} can be combined in a matrix $\Gamma := (\gamma_{ij})_{i,j=1,\dots,n}$. If we add the gains γ_{ui} to this matrix as the last row then the obtained matrix is denoted by $\overline{\Gamma}$. The expressions in the above implication motivate the following definition of a nonlinear map

$$\overline{\Gamma}_\mu : \mathbb{R}^{n+1}_+ \to \mathbb{R}^n_+ \tag{7}$$
$$\begin{bmatrix} s_1 \\ \vdots \\ s_n \\ r \end{bmatrix} \mapsto \begin{bmatrix} \mu_1(\gamma_{11}(s_1), \dots, \gamma_{1n}(s_2), \gamma_{1u}(r)) \\ \vdots \\ \mu_n(\gamma_{n1}(s_1), \dots, \gamma_{nn}(s_n), \gamma_{nu}(r)) \end{bmatrix}$$

The map Γ_μ is defined similarly. By id we will denote the identity map in an appropriate space.

Even if each subsystem Σ_i in (1) is ISS their interconnection (2) can be unstable. In the next section we recall the small gain type conditions which guarantee the ISS property of the interconnection and recall the construction of an ISS-Lyapunov function for it.

III. Small Gain Theorems

Theorem 3.1: Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$ be a gain matrix and $\mu \in \mathrm{MAF}^n_n$. If there exists an $\alpha \in \mathcal{K}_\infty$ such that the small gain condition

$$D \circ \Gamma_\mu(s) \not\geq s \quad \forall s \gneqq 0 \tag{8}$$

holds with $D := \mathrm{diag}(\mathrm{id} + \alpha)$, then there exists a continuous path $\sigma \in \mathcal{K}^n_\infty$ with

(i) σ is locally Lipschitz continuous on $(0, \infty)$ and in particular differentiable almost everywhere in $[0, \infty)$

(ii) for every compact $K \subset (0, \infty)$ there are constants $0 < c < C$ such that for all points of differentiability of σ_i and $i = 1, \dots, n$ we have

$$0 < c \leq \sigma_i'(t) \leq C\,. \tag{9}$$

(iii) $$\Gamma_\mu(\sigma(t)) < \sigma(t)\,, \quad \forall t > 0\,. \tag{10}$$

In the sequel we will call a function $\sigma \in \mathcal{K}^n_\infty$ satisfying (10) an *Ω-path with respect to* Γ_μ motivated by the existence of an invariant domain Ω to which the path parameterized by σ belongs, see [4], [6] for details concerning Ω.

Theorem 3.2: Consider the interconnected system Σ given by (1), (2) where each of the subsystems Σ_i has an ISS-Lyapunov function V_i and the corresponding gain matrix defining the map $\Gamma_m u$ as in (7). Assume there are functions $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathcal{K}^n_\infty, \varphi \in \mathcal{K}_\infty$ such that

$$\Gamma_\mu(\sigma(t), \varphi(t)) < \sigma(t)\,, \quad \forall\ t > 0 \tag{11}$$

is satisfied, then an ISS Lyapunov function for the overall system is given by

$$V(x) = \max_{i=1,\dots,n} \sigma_i^{-1}(V_i(x_i))\,, \tag{12}$$

where σ is an Ω-path corresponding to Γ_μ.

Note that by construction the Lyapunov function V is not smooth, even if the functions V_i for the subsystems are. However, theory of [1], [2] can be used in this case.

For the proofs of these theorems we refer to [4], [6], where they were given for some particular μ. Proofs with a general μ are given in [8], [11].

IV. Main results

In this section we give four examples illustrating the effectiveness of the small gain theorems of the previous section. The first of them illustrates the construction method for an ISS-Lyapunov function in the relative simple case when the interconnection consists of linear subsystems. Further we will consider the a Cohen-Grossberg neural network, an example related to formation control, and nonlinear transistor-linear resistor circuits.

A. Linear systems

Consider linear interconnected systems

$$\Sigma_i : \ \dot{x}_i = A_i x_i + \sum_{j=1}^n \Delta_{ij} x_j + B_i u_i, \quad i = 1, \dots, n \tag{13}$$

with $x_i \in \mathbb{R}^{N_i}, u_i \in \mathbb{R}^{M_i}$, and matrices A_i, B_i of appropriate dimensions. Each system Σ_i is ISS from

$(x_1^T,\dots,x_{i-1}^T,x_{i+1}^T,\dots,x_n^T,u_i^T)^T$ to x_i if and only if A_i is Hurwitz. This can be seen for example with the Lyapunov function $V_i(x_i) = x_i^T P_i x_i$, where P_i is a positive definite solution of $A_i^T P_i + P_i A_i = -Q_i$ for some positive definite matrix Q_i. It is known that there is a unique solution P_i for any given positive definite Q_i if and only if A_i is Hurwitz. In that case, along trajectories of the autonomous system

$$\dot{x}_i = A_i x_i$$

we have

$$\dot{V}_i = x_i^T P_i A_i x_i + x_i^T A_i^T P_i x_i = -x_i^T Q_i x_i \leq -c_i \|x_i\|^2$$

for $c_i := \lambda_{min}(Q_i)$, the smallest eigenvalue of Q.

For the nonautonomous system (13) we obtain

$$\begin{aligned}\dot{V}_i &= x_i^T P_i\Big(A_i x_i + \sum_{j\neq i}\Delta_{ij}x_j + B_i u_i\Big)\\ &\quad + (u_i^T B_i^T + \sum_{j\neq i} x_j^T \Delta_{ij}^T + x_i^T A_i^T) P_i x_i\\ &\leq -c_i\|x_i\|^2 + 2\|x_i\|\|P_i\|\Big(\sum_{j\neq i}\|\Delta_{ij}\|\|x_j\|\\ &\quad + \|B_i\|\|u_i\|\Big)\\ &\leq -\varepsilon c_i \|x_i\|^2, \end{aligned} \tag{14}$$

where the last inequality (14) is satisfied for some small $\varepsilon > 0$ only if

$$\|x_i\| \geq \frac{2\|P_i\|}{c_i(1-\varepsilon)}\Big(\sum_{j\neq i}\|\Delta_{ij}\|\|x_j\| + \|B_i\|\|u\|\Big) \tag{15}$$

with $u := (u_1^T,\dots,u_n^T)^T$. To write this implication in form of (6) we note that $\lambda_{\min}(P_i)|x_i|^2 \leq V_i(x_i) \leq \lambda_{\max}(P_i)|x_i|^2$. Let us denote $a_i^2 = \lambda_{\min}(P_i), b_i^2 = \lambda_{\max}(P_i)$, the inequality (15) is then satisfied if

$$V_i(x_i) \geq \Big(\sum_{j\neq i}\frac{2\|P_i\|b_i}{c_i(1-\varepsilon)}\frac{\|\Delta_{ij}\|}{a_j}\sqrt{V_j(x_j)} + \|B_i\|\|u\|\Big)^2$$

This way we see that the function V_i is an ISS-Lyapunov function for Σ_i with gains given by

$$\gamma_{ij}(s) = \Big(\sum_{j\neq i}\frac{2\|P_i\|b_i}{c_i(1-\varepsilon)}\frac{\|\Delta_{ij}\|}{a_j}\sqrt{s}$$

for $i = 1,\dots,n$, $i \neq j$, and

$$\gamma_{iu}(s) = \|B_i\|s,$$

for $i = 1,\dots,n$, and $s \geq 0$. Further we have

$$\mu_i(s,r) = \Big(\sum_{j=1}^n s_j + r\Big)^2$$

for $s \in \mathbb{R}_+^n$ and $r \in \mathbb{R}_+$. By defining $\gamma_{ii} \equiv 0$ for $i = 1,\dots,n$ we can write

$$\overline{\Gamma} = \begin{pmatrix} 0 & \gamma_{12} & \cdots & \gamma_{1n} & \gamma_{1u}\\ \gamma_{21} & \ddots & \cdots & \gamma_{2n} & \gamma_{2u}\\ \vdots & & \ddots & \vdots & \\ \gamma_{n1} & \cdots & \gamma_{nn} & 0 & \gamma_{nu}\end{pmatrix}$$

and have

$$\overline{\Gamma}_\mu(s,r) = \begin{pmatrix}\Big(\frac{2\|P_1\|b_1}{c_1(1-\varepsilon)}\Big)^2\Big(\sum_j \frac{\|\Delta_{1j}\|}{a_1}\sqrt{s_j} + \|B_1\|r\Big)^2\\ \vdots\\ \Big(\frac{2\|P_n\|b_n}{c_n(1-\varepsilon)}\Big)^2\Big(\sum_j \frac{\|\Delta_{nj}\|}{a_n}\sqrt{s_j} + \|B_n\|r\Big)^2\end{pmatrix}. \tag{16}$$

Interestingly, the choice of quadratic Lyapunov functions for the subsystems naturally leads to a nonlinear mapping $\overline{\Gamma}$.

Proposition 4.1: Let each Σ_i in (13) be ISS with a quadratic ISS-Lyapunov function V_i, so that the corresponding operator Γ_μ can taken to be as in (16). If the spectral radius of the assocated matrix

$$G = \Big(\tfrac{2\|P_i\|b_i}{c_i(1-\varepsilon)}\tfrac{\|\Delta_{ij}\|}{a_j}\Big)_{ij} \tag{17}$$

is less than 1, then the interconnection

$$\Sigma: \quad \dot{x} = (A+\Delta)x + Bu$$

is ISS and its (nonsmooth) ISS-Lyapunov function can be taken as $V(x) = \max_i \frac{1}{\hat{s}_i}x_i^\top P_i x_i$ for some positive vector $\hat{s} \in \mathbb{R}_+^n$.

Proof: If the spectral radius of G is less than one, then there exists a positive vector $\tilde{s}$ satisfying $G\tilde{s} < \tilde{s}$: Just add a small $\delta > 0$ to every entry of G, so that the spectral radius $\rho(\tilde{G})$ of $\tilde{G} = G + \delta$ is still less than one, due to continuity of the spectrum. Then there exists a Perron-Frobenius vector $\tilde{s}$ such that $G\tilde{s} < \tilde{G}\tilde{s} = \rho(\tilde{G})\tilde{s} < \tilde{s}$.

Now define $\hat{s}$ by $\hat{s}_i = \tilde{s}_i^2$ for $i = 1,\dots,n$. We claim that the straight half-line spanned by this vector $\hat{s}$ in the positive orthant is an Ω-path for Γ_μ.

Indeed, for all $t > 0$ we have

$$\begin{aligned}\Gamma_\mu(\hat{s}t) &= \left(\sum_j \frac{2\|P_i\|b_i}{c_i(1-\varepsilon)}\frac{\|\Delta_{ij}\|}{a_j}\sqrt{\hat{s}_j t}\right)^2\\ &= \left(\sum_j \frac{2\|P_i\|b_i}{c_i(1-\varepsilon)}\frac{\|\Delta_{ij}\|}{a_j}\tilde{s}_j\right)^2 \cdot t\\ &< (\tilde{s}_i)^2 t = \hat{s}_i t.\end{aligned}$$

By Theorem 3.2 an ISS-Lyapunov function can be taken as $V(x) = \max_i \frac{1}{\hat{s}_i}x_i^\top P_i x_i$. ∎

B. Neural networks

Consider a Cohen-Grossberg neural network, see [16], e.g., given by

$$\dot{x}_i(t) = -a_i(x_i(t))\Big(b_i(x_i(t)) - \sum_{j=1}^n t_{ij}s_j(x_j(t)) + J_i\Big), \tag{18}$$

$i = 1,\dots,n,\ n \geq 2$, where x_i denotes the state of the ith neuron, a_i is strictly positive amplification function, b_i typically has the same sign as x_i and is assumed to satisfy $|b_i(x_i)| > \tilde{b}_i(|x_i|)$ for some $\tilde{b}_i \in \mathcal{K}$, the activation function s_i is typically assumed to be sigmoid. The matrix $T = (t_{ij})_{i,j=1,\dots,n}$ describes the interconnection of neurons in the network and J_i is a given constant input from outside. However for our consideration we allow J_i to be arbitrary measurable function in L_∞.

Note that for any sigmoid function there exists a $\gamma_i \in \mathcal{K}$ such that $|s_i(x_i)| < \gamma_i(|x_i|)$, following [16] we assume $0 < \underline{\alpha}_i < a_i(x_i) < \overline{\alpha}_i$.

Define $V_i(x_i) = |x_i|$ then each subsystem is ISS since the following implication holds

$$|x_i| > \tilde{b}_i^{-1}\left(\frac{\overline{\alpha}_i}{\underline{\alpha}_i - \varepsilon}\left(\sum_{j=1}^{n} |t_{ij}|\gamma_j(|x_j|) + |J_i|\right)\right) \implies$$

$$\dot{V}_i = -a_i(x_i)\Big(|b_i(x_i)| - \operatorname{sign} x_i \sum_{j=1}^{n} t_{ij} s_j(x_j) + \operatorname{sign} x_i J_i\Big)$$

$< -\varepsilon |b_i(x)|$.

In this case we have $\mu_i(s,r) = \tilde{b}_i^{-1}(s_1 + \cdots + s_n + r)$ and $\gamma_{ij} = \frac{\overline{\alpha}_i |t_{ij}|}{\underline{\alpha}_i - \varepsilon}\gamma_j(|x_j|)$, $\gamma_{iu} = \frac{\overline{\alpha}_i \mathrm{id}}{\underline{\alpha}_i - \varepsilon}$. Note that so far we have not imposed any restrictions on the coefficients t_{ij}. Moreover the assumptions imposed on a_i, b_i, s_i are essentially milder then in [16]. However to obtain the ISS property of the network we need to require more. The small gain condition can be used for this purpose. It will impose some restrictions on the coupling terms $t_{ij}s(x_j)$. It follows then:

Theorem 4.2: Let Γ_μ be given by γ_{ij} and μ_i, $i, j = 1, \ldots, n$ calculated above for the Cohen-Grossberg neural network (18) satisfy the small gain condition (8). Then this network is ISS from $(J_1, \ldots, J_n)^T$ to x.

Remark 4.3: In [16] the authors have proven that there exists a unique equilibrium point for the network and given constant external inputs. They have also proved the exponential stability of this equilibrium. We have considered arbitrary external inputs in the network and proved the ISS property for the interconnection.

C. Formation control

In [15] formations of vehicles on the plane as in Figure 1 have been considered. Using feedback linearization local controllers have been designed that render the formation error between two consecutive vehicles input-to-state stable with respect to the up-link formation error. Since cascades of ISS systems are ISS it could be shown that cascades of vehicles are “leader-to-formation” stable (ISS).

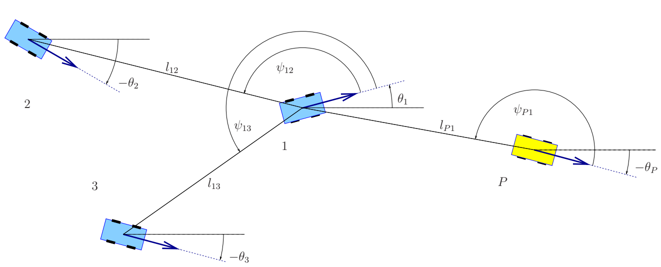

Fig. 1. A formation of four vehicles (1–3) following a designated leader (P).

Due to the converse ISS Lyapunov theorems, it is clear that there has to exist an ISS Lyapunov function for the entire formation error; here we show how to find it.

The dynamics for each vehicle i in Figure 1 is given by

$$\dot{x}_i = v_i \cos\theta_i, \quad \dot{y}_i = v_i \sin\theta_i, \quad \dot{\theta}_i = w_i, \tag{19}$$

(x_i, y_i, θ_i) being the absolute position and orientation of the ith vehicle. The control inputs (v_i, w_i) are the translational and rotational velocity. The separation distance between two consecutive vehicles, a leader i and a follower j, is denoted by l_{ij}, and the relative bearing between them is ψ_{ij}. For both values constant specification parameters l_{ij}^d, ψ_{ij}^d are fixed and describe the formation. The formation error is

$$h_{ij} := (\tilde{l}_{ij}, \tilde{\psi}_{ij})^\top := (l_{ij}^d - l_{ij}, \psi_{ij}^d - \psi_{ij})^\top. \tag{20}$$

The control objective is to drive h_{ij} to 0. In [15] a controller is proposed that renders $\tilde{z}_{ij} = (\tilde{l}_{ij}, \tilde{\psi}_{ij}, \varphi_{ij})^\top$ ISS with respect to $\tilde{z}_{ki}$, there k denotes the number of the vehicle in front of vehicle i. It is shown that the Lyapunov function candidate

$$V_{ij}(\tilde{z}_{ij}) = \frac{1}{2k_1^j}\|\tilde{l}_{ij}\|^2 + \frac{1}{2k_2^j}\|\tilde{\psi}_{ij}\|^2 \tag{21}$$

satisfies the implication

$$\|\tilde{z}_{ij}\| \geq \frac{\max\{k_1^i, k_2^i\}(d + l_{ki}^d + \|\tilde{z}_{ki}\|)\|\tilde{z}_{ki}\|}{(1-\varepsilon)d\min\{k_1^j, k_2^j\}} \implies \dot{V}_{ij} \leq -\varepsilon\|\tilde{z}_{ij}\|^2. \tag{22}$$

Implication (22) can be rewritten as

$$\|\tilde{z}_{ij}\| \geq \frac{\max\{k_1^i, k_2^i\}(d + l_{ki}^d + V_{ki}(\tilde{z}_{ki}))V_{ki}(\tilde{z}_{ki})}{(1-\varepsilon)d\min\{k_1^j, k_2^j\}} \implies \dot{V}_{ij} \leq -\varepsilon\|\tilde{z}_{ij}\|^2. \tag{23}$$

which is in the form of (6). For a cascade formation of n vehicles the gain matrix takes the form

$$\Gamma = \begin{pmatrix} 0 & \ldots & \ldots & 0 \\ \gamma_{21} & 0 & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \gamma_{n1} & \ldots & \gamma_{n,n-1} & 0 \end{pmatrix} \tag{24}$$

where γ_{ij} is given according to (23) and μ_i is summation for $i = 1, \ldots, n$. An Ω-path $\sigma = (\sigma_1, \ldots, \sigma_n)^\top \in \mathcal{K}_\infty^n$ can now iteratively constructed as follows: Pick $\sigma_1 = \mathrm{id} \in \mathcal{K}_\infty$. Iteratively, for $i = 2, \ldots, n$, choose $\sigma_i \in \mathcal{K}_\infty$ such that $\sum_{j=1}^{i-1} \gamma_{ij} \circ \sigma_j < \sigma_i$. Clearly that is always possible, as a sum of $\mathcal{K}_\infty$ functions can always be bounded from above by a $\mathcal{K}_\infty$ function.

The ISS Lyapunov function for the formation error is now given according to formula (12).

D. Transistor networks

In [10] a nonlinear transistor-linear resistor model is considered, which is characterized by the set of equations

$$\dot{z}_i + A_i f_i(z_i) + B_i g_i(z_i) = b_i(t), \quad i = 1, \ldots, n, \tag{25}$$

where $z_i \in \mathbb{R}^{N_i}, A_i = (a_{kl}^i), B_i = (b_{kj}^i)$ are constant square matrices with $a_{kk}^i > 0$ and $b_{kk}^i > 0$, where $f_i, g_i : \mathbb{R}^{N_i} \to \mathbb{R}^{N_i}$ are continuously differentiable in z_i, satisfy

$f_i(z_i) = 0$ and $g_i(z_i) = 0$ if and only if $z_i = 0$, and further it may be assumed that in fact $f_k^i(z_i) = f_k^i(z_k^i)$ and $g_k^i(z_i) = g_k^i(z_k^i)$ where $f_i(z_i) = (f_1^i(z_i), \dots, f_{N_i}^i(z_i))^\top$, $g_i(z_i) = (g_1^i(z_i), \dots, g_{N_i}^i(z_i))^\top$, and $z_i = (z_1^i, \dots, z_{N_i}^i)^\top$. As in [12] it is assumed that $f_k^i(z_k^i)/z_k^i \geq \delta > 0$ and $g_k^i(z_k^i)/z_k^i \geq \delta$ for all $z_k^i \neq 0$ and that $\partial/\partial z_k^i f_k^i(0) \geq \delta$ and $\partial/\partial z_k^i f_k^i(0) \geq \delta$. As in [10] we content ourselves to the case

$$b_i(t) \equiv 0, \quad i = 1, \dots, n.$$

Interconnection between the subsystems (25) is described via terms $C_{ij} g_j(z_j)$, where C_{ij} are constant matrices of appropriate dimensions, leading to subsystem dynamics

$$\dot{z}_i + A_i f_i(z_i) + B_i g_i(z_i) + \sum_{j \neq i} C_{ij} g_j(z_j) = 0, \quad i = 1, \dots, n. \tag{26}$$

This may be written as

$$\dot{z} + Af(z) + Bg(z) = 0, \tag{27}$$

where $z^\top = (z_1^\top, \dots, z_n^\top)$, $f^\top = (f_1^\top, \dots, f_n^\top)$, $g^\top = (g_1^\top, \dots, g_n^\top)$, A accommodates the matrices A_i on the diagonal, and B accommodates the matrices B_i and C_{ij} in the obvious manner.

Due to the nonlinear terms f_i a quadratic Lyapunov function $V_i(z_i) = z^\top P_i z_i$ with $PA_i + A_i^\top P_i = -Q_i$ for some positive definite matrix Q_i bears some difficulties when it comes to consider the interconnection structure and to derive gains between the subsystems. Instead, we decompose system (27) into its scalar subsystems

$$\tilde{z}_i = -(a_{ii} + b_{ii}) \tilde{f}_i(x_i) + \sum_{j \neq i} (a_{ij} \tilde{f}_j(\tilde{z}_j) + b_{ij} \tilde{g}_j(\tilde{z}_j)), \tag{28}$$

for $i = 1, \dots, N = \sum_j N_j$, $A = (a_{ij})$, $B = (b_{ij})$ and where $\tilde{f}_i$ and $\tilde{g}_i$ denote the component functions of f and g, respectively, and z is decomposed into $\tilde{z}_1, \dots, \tilde{z}_N$.

Using a small $\varepsilon > 0$, standard estimates, and the Lyapunov function candidates $V_i(\tilde{z}_i) = \frac{1}{2}\tilde{z}_i^2$, we arrive at

$$\dot{V}_i \leq -(a_{ii} + b_{ii}) \delta \varepsilon \tilde{z}_i^2 \tag{29}$$

if

$$|\tilde{z}_i| \geq \frac{1}{(1-\varepsilon)\delta} \sum_{j \neq i} \tilde{\gamma}_{ij}(|z_j|) \tag{30}$$

where

$$\begin{aligned} \tilde{\gamma}_{ij}(s) = & \frac{|a_{ij}|}{a_{ii}} \max\{|\tilde{f}_j(s)|, |\tilde{f}_j(-s)|\} \\ & + \frac{|b_{ij}|}{b_{ii}} \max\{|\tilde{g}_j(s)|, |\tilde{g}_j(-s)|\}. \end{aligned} \tag{31}$$

Now we note that due to the properties of the functions $\tilde{f}_i$ and $\tilde{g}_i$, the function $\tilde{\gamma}_{ij}$ is of class $\mathcal{K}_\infty$ or equals constantly zero. Hence for the functions γ_{ij} given by $\gamma_{ij}(s) = \frac{1}{(1-\varepsilon)\delta} \tilde{\gamma}_{ij}(\sqrt{2s})$, we have

$$\begin{aligned} & V_i(\tilde{z}_i) \geq \frac{1}{2} \Big(\sum_{j \neq i} \gamma_{ij}(V_j(z_j)) \Big)^2 \\ & \implies \dot{V}_i \leq -2(a_{ii} + b_{ii}) \delta \varepsilon V_i(\tilde{i}) \end{aligned} \tag{32}$$

i.e., the defining implication form (6) for an ISS Lyapunov function.

Now, if the small gain condition is satisfied for $\Gamma = (\gamma_{ij})$ and $\mu_i(s) = \frac{1}{2} \left(\sum_j s_j \right)^2$, then the trivial solution of (27) is globally asymptotically stable, and a Lyapunov function is given by (12).

V. Conclusions

In this paper we have considered several ISS systems with the aim to investigate the stability properties of their interconnections. The generalized small gain results were used for this purpose. This paper shows how these results may be applied to prove the ISS property of interconnections of nonlinear systems.

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