0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in "Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.1

Manuscript submitted to
AIMS' Journals
Volume **00**, Number **0**, Xxxx **XXXX**

xx/xxxx-xxxx

pp. **000–000**

SEPARABLE LYAPUNOV FUNCTIONS FOR MONOTONE SYSTEMS: CONSTRUCTIONS AND LIMITATIONS

GUNTHER DIRR, HIROSHI ITO, ANDERS RANTZER, AND BJÖRN S. RÜFFER

Abstract. For monotone systems evolving on the positive orthant of $\mathbb{R}^n_+$ two types of Lyapunov functions are considered: Sum- and max-separable Lyapunov functions. One can be written as a sum, the other as a maximum of functions of scalar arguments. Several constructive existence results for both types are given. Notably, one construction provides a max-separable Lyapunov function that is defined at least on an arbitrarily large compact set, based on little more than the knowledge about one trajectory. Another construction for a class of planar systems yields a global sum-separable Lyapunov function, provided the right hand side satisfies a small-gain type condition. A number of examples demonstrate these methods and shed light on the relation between the shape of sublevel sets and the right hand side of the system equation. Negative examples show that there are indeed globally asymptotically stable systems that do not admit either type of Lyapunov function.

1. Introduction.

In this paper we consider dynamical systems defined on $\mathbb{R}^n_+ := [0,\infty)^n$ via the differential equation

$$\dot{x} = f(x) \tag{1}$$

with $f\colon \mathbb{R}^n_+ \to \mathbb{R}^n$ locally Lipschitz continuous and $f(0) = 0$. Throughout the paper we assume that the system is monotone, i.e., that the *ordering of solutions principle*

$$x \le y \implies \varphi(t,x) \le \varphi(t,y)$$

holds for solutions φ for all initial conditions $x, y \in \mathbb{R}^n_+$ and all $t \ge 0$ where both solutions exist. Monotonicity and the equilibrium at the origin imply that system (1) leaves $\mathbb{R}^n_+$ invariant in forward time. (For a monotone system with equilibrium in x^0, the set $\{y \in \mathbb{R}^n : x^0 \le y\}$ must be positively invariant, so we might as well shift coordinates to get $x^0 = 0$.)

Monotone systems appear in a variety of real-world scenarios, such as chemical reaction networks [9], gene expression [28] and general systems biology [41], as well as traffic networks [7].

We are interested in the asymptotic stability of the origin and the characterization of this asymptotic stability by means of a Lyapunov function $V\colon \mathbb{R}^n_+ \to \mathbb{R}_+$, i.e., a positive definite function that decreases along trajectories in a neighborhood of the origin. If V is continuously differentiable then for points $x \ne 0$ a sufficient

This publications subsumes material presented at and published in the proceedings of the IEEE Conferences on Decision and Control in 2013 [31] and 2014 [24].

Hiroshi Ito has been supported by the Grant-in-Aid for Scientific Research (C) of JSPS under grant 26420422.

Anders Rantzer is supported by the LCCC Linnaeus Center and the eLLIIT Excellence Center at Lund University.

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in "Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.2

condition for this decrease is given by the condition

$$\nabla V(x)f(x) = \frac{d}{dt}V\big(\varphi(t,x)\big)\Big|_{t=0} < 0. \tag{2}$$

The differentiability requirement can be weakened to locally Lipschitz continuity by resorting to Dini derivatives [44] or to appropriate subgradient methods [6], amounting essentially to require (2) to hold at all points of differentiability, cf. Corollary 1.

Of particular interest to us is the case where V can be separated into n individual functions of one scalar argument, either as

$$V(x) = \sum_{i=1}^{n} V_i(x_i) \tag{3}$$

or as

$$V(x) = \max_{i=1,\dots,n} V_i(x_i). \tag{4}$$

In the first case we call V a *sum-separable* Lyapunov function and in the latter a *max-separable* Lyapunov function.

Separable Lyapunov functions have been used widely in the existing literature on stability analysis of monotone systems, although usually they have not been named "separable". Recent examples include [7, Lemma 2] as well as [28, Theorem 2]. In both results it is shown that the l_1-distance is a sum-separable Lyapunov function for the systems under consideration.

Our own interest stems from applications in nonlinear distributed control and stability analysis of large-scale systems (e.g.[21, 20, 22, 23, 8, 36, 29, 30]). By way of an example, separable Lyapunov functions appear in the construction of Lyapunov functions for composite systems from Lyapunov functions for individual subsystems as follows. In applications a composite system

$$\dot{y} = F(y), \quad F\colon \mathbb{R}^N \to \mathbb{R}^N \tag{5}$$

may appear as an interconnection of $n \geq 2$ subsystems,

$$\dot{y}_i = F_i(y_1, \dots, y_n),$$

with $F_i\colon \mathbb{R}^{N_1} \times \dots \times \mathbb{R}^{N_n} = \mathbb{R}^N \to \mathbb{R}^{N_i}$, $i = 1, \dots, n$. It is usually assumed that every such subsystem is endowed with a suitable Lyapunov function $V_i\colon \mathbb{R}^{N_i} \to \mathbb{R}_+$ that quantifies the subsystem's stability with respect to input from other subsystems, e.g., via

$$\nabla V_i(y_i)F_i(y_1, \dots, y_n) \leq -\alpha_i\big(V_i(y_i)\big) + \sum_{j\neq i} \gamma_{ij}\big(V_j(y_j)\big) \tag{6}$$

for suitable positive definite scalar functions α_i and non-negative and non-decreasing scalar functions γ_{ij}. More precisely, one could assume that every subsystem is input-to-state stable (ISS) [39] with an ISS Lyapunov function V_i [42]. For this case it was shown in [25, 8] that under suitable conditions $V(x) = \max_i \sigma_i^{-1}\big(V_i(x_i)\big)$ is a Lyapunov function for the composite system (5), where the functions σ_i are appropriate scaling functions. Clearly, this composite Lyapunov function is of the form (4).

However, when subsystems are allowed to satisfy relaxed stability assumptions, e.g, they are only assumed to be *integral* input-to-state stable (iISS) [40], then it was found that the same construction from [25, 8] is too restrictive and essentially implies that the subsystems must all be ISS [22]. Instead, a construction based on (3) has been used successfully at different occasions, e.g. in [21, 20, 22, 23].

Both of these constructions are related to monotone systems, as the respective stability conditions for large-scale systems (6) can always be translated into a stability condition on a lower-dimensional, monotone comparison system [33, 36]. In

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in "Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.3

our example, the resulting lower dimensional system is of the form (1) with f given by

$$f(x) = \begin{pmatrix} -\alpha_1(x_1) + \sum_j \gamma_{1j}(x_j) \\ \vdots \\ -\alpha_n(x_n) + \sum_j \gamma_{nj}(x_j) \end{pmatrix} \tag{7}$$

and the ordering of solutions is guaranteed as long as the functions γ_{ij} are non-decreasing, see [36].

A natural question thus is: If Lyapunov functions of the form (3) can seem to handle "more general" types of interconnections of stable subsystems, is the set of monotone (comparison) systems admitting such a Lyapunov function bigger than the class of systems only admitting a Lyapunov function of the form (4)?

A separate interest is to estimate the region of attraction of the origin for system (1), provided the attraction is not global to begin with. An estimate of this region for the monotone comparison system based on (7) can be translated to an estimate of the region of attraction for the composite system (5).

In this paper, we give several constructions for local and global separable Lyapunov functions. We discuss limitations of these Lyapunov functions for estimating regions of attraction as well as the implications that the existence of either kind of Lyapunov function has on the dynamics of the monotone, respectively, original dynamics. In addition, we construct specific monotone systems that do admit one type of global separable Lyapunov function but not the other or that do not admit any global separable Lyapunov function.

This paper is organized as follows: In the next section we will introduce some relevant notation and definitions related to monotone systems and Lyapunov stability.

In Section 3, we show constructions of sum- and max-separable Lyapunov functions. Starting with local results(§ 3.1.1), we also provide a semi-global result showing that on a compact domain one can always find a max-separable Lyapunov function by using little more than the knowledge about one trajectory. However, our construction may lead to discontinuous Lyapunov functions, as we detail in an example. Under additional assumptions on the system, the same construction also yields a global Lyapunov function(§ 3.1.2). In Section 3.1.3, we discuss limitations that the existence of max-separable Lyapunov functions impose on the system dynamics in view of their sublevel sets. These limitations are growth conditions on the vector field f. If f arises as a comparison system then these limitations can be interpreted as fundamental restrictions on the stability classes of systems and topologies of the large-scale interconnections.

Sum-separable Lyapunov functions are considered in Section 3.2. After presenting a local, linearization-based result in Section 3.2.1, we turn our attention to planar systems, for which we provide a global construction in Section 3.2.2.

All constructions are supported by examples that demonstrate how the shape of the sublevel sets is in correspondence with the sign-pattern of the right hand side of the system dynamics.

Finally, in Section 4 we give examples of systems that do not admit separable Lyapunov functions of one type but of the other, before we give a brief summary in Section 5.

2. **Preliminaries.**

In this section, we recall a number of system theoretic concepts and formalize the statements given in the introduction.

2.1. **Kamke-Müller conditions and monotonicity.**

We consider $\mathbb{R}^n$ equipped with the component-wise partial order, which we denote by $x \leq y$ if $x_i \leq y_i$ for all i, $x < y$ if $x \leq y$ but $x \neq y$, and $x \ll y$ if $x_i < y_i$ for all i. Given a bounded set $X \subset \mathbb{R}^n$, by $\sup X$ we denote the smallest upper bound with respect to this partial order, i.e., $\sup X = (\sup_{x\in X} x_1, \dots, \sup_{x\in X} x_n)$. A map $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is *monotone* if $x \le y$ implies $F(x) \le F(y)$. For a partially ordered set $A \subset \mathbb{R}^n$, we define $A_+ := \{a \in A\colon a \ge 0\}$. Moreover, we denote the n-vector $(1, \dots, 1)$ as well as the $n \times n$ matrix whose entries are all equal to 1 both by $\mathbf{1}$.

In this work we consider systems of the form (1). The local Lipschitz continuity assumption guarantees the local existence and uniqueness of solutions $\varphi(t, x)$ on some maximal open time interval containing 0. Where they exist, the solutions satisfy $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$ and $\varphi(0, x) = x$.

Throughout this paper we will assume that system (1) is monotone, i.e., $x \le y$ implies $\varphi(t, x) \le \varphi(t, y)$ for all $t \in \mathbb{R}_+$. This holds if and only if f satisfies the *Kamke-Müller conditions*[1], cf. [27, 37],

$$x \le y \text{ and } x_i = y_i \implies f_i(x) \le f_i(y). \tag{8}$$

If the vector field f is continuously differentiable then this condition is equivalent to the requirement that the off-diagonal entries of the Jacobian matrix $Jf(x)$ are nonnegative for every x—and in this case the system is called *cooperative*, cf. the seminal works by Hirsch [12, 13, 14, 16, 15, 17], the textbook [37], as well as the more recent review article [18].

2.2. **Invariant sets.** We define two sets where trajectories are non-increasing, respectively, strictly decreasing in each component for system (1). These have been termed *decay sets* [32] and are defined as

$$\begin{aligned} \Psi &:= \left\{x \in \mathbb{R}^n_+ \colon\ f(x) \le 0\right\} \qquad \text{and} \\ \Omega &:= \left\{x \in \mathbb{R}^n_+ \colon\ f(x) \ll 0\right\}. \end{aligned} \tag{9}$$

The following result is fundamental.

Lemma 2.1 ([36])**.** *If the origin is attractive with respect to* (1)*, then it is also stable. In the region of attraction it holds that $f(x) \not\ge 0$ for $x \ne 0$ and that the sets Ψ and Ω are non-empty in the sense that $\Psi \cap S_r \ne \emptyset$ and $\Omega \cap S_r \ne \emptyset$ for every $r > 0$ such that $S_r := \{x \in \mathbb{R}^n_+ \colon \|x\|_1 = r\}$ is contained in the region of attraction. Moreover, the sets Ψ and Ω are positively invariant.*

2.3. **Stability.** The origin is *asymptotically stable* if it is attractive and stable in the sense of Lyapunov. It is *globally asymptotically stable* if it is asymptotically stable and its region of attraction is the entire $\mathbb{R}^n_+$.

In this paper we will frequently drop the explicit reference to the origin and just talk about stability or attractivity. A subset $\mathcal{B} \subseteq \mathbb{R}^n_+$ will be called a *domain of attraction* if $\lim_{t\to\infty} \varphi(t, x) = 0$ holds for all $x \in \mathcal{B}$.

2.4. **Comparison functions.** A function $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{P}$ and written as $\rho \in \mathcal{P}$ if it is continuous and positive definite, i.e., $\rho(t) \ge 0$ for all $t \ge 0$ and $\rho(s) = 0$ if and only if $s = 0$.

A class $\mathcal{P}$ function is of class $\mathcal{K}$ if it is strictly increasing. A class $\mathcal{K}$ function is of class $\mathcal{K}_\infty$ if it is unbounded.

Given $\omega \in \mathcal{K}$, the function $\omega^{\ominus}\colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is defined as $\omega^{\ominus}(s) = \sup\{v \in \mathbb{R}_+\colon\ s \ge \omega(v)\}$. By definition, $\omega^{\ominus}(s) = \infty$ for $s \ge \lim_{\tau\to\infty} \omega(\tau)$, and $\omega^{\ominus}(s) = \omega^{-1}(s)$ elsewhere. A function $\omega \in \mathcal{K}$ is extended to $\omega\colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ by $\omega(\infty) := \sup_{s\in\mathbb{R}_+} \omega(s)$.

[1]Functions satisfying (8) are also referred to as *quasi-monotone nondecreasing* or *type K*.

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in "Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.5

2.5. **Lyapunov functions and their derivatives with respect to time.** In the literature Lyapunov functions are usually assumed to be at least differentiable, if not smooth. For technical reasons, however, we consider a wider class of functions in this paper.

Let $\mathcal{D} \subseteq \mathbb{R}^n_+$ be an open neighborhood of the origin. Recall that a function $V : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}_+$ is called *lower-semicontinuous* if

$$\liminf_{y \to x} V(y) \geq V(x)$$

holds for all $x \in \mathcal{D}$. A lower-semicontinuous function $V : \mathcal{D} \to \mathbb{R}_+$ is called a *strict Lyapunov function* for (1), if it satisfies the following conditions:

1. There exist α_1 and $\alpha_2 \in \mathcal{K}_\infty$ such that
$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{10}$$
for all $x \in \mathcal{D}$
2. V is strictly decreasing along trajectories of (1) starting from $x \in \mathcal{D} \setminus \{0\}$, i.e., for all $x \in \mathcal{D} \setminus \{0\}$ and all $\tau > 0$ such that $\varphi(t, x) \in \mathcal{D}$ for all $t \in [0, \tau]$ the function $t \mapsto V(\varphi(t, x))$ is strictly decreasing on $[0, \tau]$.

Note that condition (10) guarantees continuity of V at the origin. Furthermore, when $\mathcal{D}$ is unbounded, (10) implies that V is radially unbounded, that is, $V(x) \to \infty$ for $\|x\| \to \infty$.

Theorem 2.2. *Let $\mathcal{D} \subset \mathbb{R}^n_+$ be an open neighborhood of the origin and let $V : \mathcal{D} \to \mathbb{R}_+$ be a Lyapunov function in the above sense. Moreover, let V satisfy additionally one of the following assumptions:*

1. *V is continuous.*
2. *The* contingent derivative *V^* of V is negative definite, i.e., there exists a continuous, positive definite function $\alpha_3 : \mathbb{R}_+ \to \mathbb{R}_+$, s.t.*
$$V^\star(x) := \liminf_{\tau \searrow 0, \|y\| \to 0} \frac{V(x + \tau f(x) + \tau y) - V(x)}{\tau} \leq -\alpha_3(\|x\|) \tag{11}$$
for all $x \in \mathcal{D}$.

Then the origin is asymptotically stable for the monotone system (1).

Sketch of the proof. If $V : \mathcal{D} \to \mathbb{R}_+$ is continuous, the above statement is a standard result from Lyapunov Theory [10, Theorem 3.2.7]. If V is only lower-semicontinuous, [5, Theorem IX.2.1] implies that V is strictly decreasing along trajectories and that the origin is attractive. Stability is a consequence of monotonicity and attractivity and follows from Lemma 2.1. □

Remark 1. In general, one has the following relation between the contingent derivative V^* and the Dini derivative $\dot{V}$ of V along trajectories of (1):

$$V^\star(x) \leq \dot{V}(x) := \liminf_{h \searrow 0} \frac{V(\varphi(t + h, x)) - V(x)}{h}.$$

Therefore, $\dot{V}(x) < 0$ implies $V^\star(x) < 0$, see, e.g., [43]. It is shown in [5, Theorem IX.1.6] that (11) is a sufficient condition for V to be strictly decreasing along trajectories of system (1).

Corollary 1. *Let $\mathcal{D} \subset \mathbb{R}^n_+$ be an open neighborhood of the origin, $V : \mathcal{D} \to \mathbb{R}_+$ a locally Lipschitz continuous function and $\alpha_3 : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and positive definite such that*

$$\nabla V(x) f(x) \leq -\alpha_3(\|x\|) \tag{12}$$

for all points $x \in D$, where V is differentiable. Then V is strictly decreasing along trajectories of (1) *emanating from $x \in \mathcal{D} \setminus \{0\}$ and thus the origin is asymptotically stable for system* (1).

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in p.6
"Discrete and Continuous Dynamical Systems - Series B (DCDS-B)"

Sketch of the proof. The claim follows from the characterization of Clarke's generalized gradient using Rademacher's Theorem [6] and an application of [3, Theorem 2.1]. □

2.6. **Sublevel sets.** Given an open neighborhood $\mathcal{D} \subseteq \mathbb{R}^n_+$ of the origin, $l \in \mathbb{R}_+$, and a positive definite function $V : \mathcal{D} \to \mathbb{R}_+$, we define its *sublevel sets*

$$L(l) := \{x \in \mathcal{D} \colon V(x) \leq l\}. \tag{13}$$

In the case of unbounded $\mathcal{D}$, the sets $L(l)$ are bounded for every $l > 0$ if and only if V is radially unbounded.

If for a given $l > 0$ the sublevel set $L(l)$ is compactly embedded in $\mathcal{D}$ and V nonincreasing along trajectories of (1) then $L(l)$ is positively invariant. If, moreover, V satisfies the conditions of Theorem 2.2 then $L(l)$ is contained in the domain of attraction.

2.7. **Input-to-state stability and integral input-to-state stability.** A system $\dot{x} = g(x, u)$ with $g : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ satisfying standard assumptions on local existence and uniqueness of solutions is *input-to-state stable from u to x* (ISS), cf. [38, 39], if there exists a smooth function $V : \mathbb{R}^N \to \mathbb{R}_+$ satisfying (10) and functions $\alpha, \gamma \in \mathcal{K}_\infty$ such that

$$\nabla V(x) f(x, u) \leq -\alpha(V(x)) + \gamma(\|u\|) \tag{14}$$

for all $x \in \mathbb{R}^N$ and all $u \in \mathbb{R}^M$.

The system is *integral* input-to-state stable (iISS), cf. [40], if V as above satisfies (10) and there exist functions $\alpha \in \mathcal{P}$, $\gamma \in \mathcal{K}_\infty$ such that (14) holds. In fact, it is known that ISS implies iISS (and proved in [40, 2]).

3. **Constructions and limitations.** Now we can detail several methods to construct separable Lyapunov functions. The constructions will be demonstrated in several examples based on the class of systems defined by

$$\dot{x}_1 = -x_1 + x_1 x_2 \tag{15}$$

$$\dot{x}_2 = -2x_2 - x_2^2 + 2g(x_1) + g(x_1)^2, \tag{16}$$

where g is a class $\mathcal{K}$ function satisfying

$$\overline{g} := \lim_{s \to \infty} g(s) < 1. \tag{17}$$

System (15)–(16) is monotone, and the set Ω in (9) is computed as

$$\Omega = \{x \in \mathbb{R}^2_+ \colon g(x_1) < x_2 < 1\}.$$

Clearly, this set Ω divides $(0, \infty)^2$ into two disjoint sets. It is shown, e.g., in [1, 20, 21] that this fact together with (17) guarantees that the origin is globally asymptotically stable for system (15)–(16). The basic argument is that the positively invariant set Ω must be reached in finite time from any initial condition, which can be seen from geometric considerations.

3.1. **Max-separable Lyapunov functions.**

3.1.1. *Local max-separable Lyapunov functions.* We start with a local result: One can always find a max-separable Lyapunov function in a neighborhood of the origin.

Theorem 3.1. *Assume that f is continuously differentiable at the origin and that $Df(0)$ is Hurwitz. Then there exists a Lipschitz continuous max-separable Lyapunov function for (1) in a neighborhood of the origin.*

Proof. The Jacobian $A = Df(0)$ is a Metzler matrix with negative spectral abscissa. For a small $\epsilon > 0$ the matrix $\tilde{A} := A + \epsilon\mathbf{1}$ is, by the continuity of the spectrum, again Metzler and has a negative spectral abscissa, but all off-diagonal entries of A are positive. By the Perron-Frobenius Theorem (see, e.g., [4]) there exists a right-eigenvector $\sigma \gg 0$ of $\tilde{A}$ such that $A\sigma \ll \tilde{A}\sigma = a\sigma \ll 0$, where $a < 0$ is the spectral abscissa of $\tilde{A}$. The function

$$V(x) := \max_i \sigma_i^{-1} x_i \tag{18}$$

is positive definite on $\mathbb{R}^n_+$. By construction V is Lipschitz continuous. Along trajectories of the linearized system

$$\dot{x} = Ax$$

the function V is strictly decreasing, as we show next. First we note that by definition of V we have $x \leq V(x)\sigma$. Furthermore, at points of differentiability of V we have that $V(x) = \sigma_i^{-1} x_i > 0$ for some index i. Hence $\dot{V}(x) = \sigma_i^{-1} e_i^\top Ax \leq \sigma_i^{-1} e_i^\top AV(x)\sigma < \sigma_i^{-1} a e_i^\top V(x)\sigma = aV(x) < 0$. It follows that V is a Lyapunov function in the sense of Corollary 1.

A standard linearization argument involving the Taylor series of f around 0 establishes that V is also a Lyapunov function for the nonlinear system in a neighborhood of the origin. ☐

The drawback of the previous result is of course that in general it is difficult to tell *a priori* what the maximal sublevel set of V is, on which $\dot{V}$ is negative definite. This obstacle is overcome by our next construction:

One can always find a max-separable Lyapunov function on arbitrary compact, positively invariant sets in the domain of attraction by exploiting essentially only the knowledge about one trajectory. Related approaches of verifying stability and robustness of monotone systems using a single trajectory only have also been pursued in [34, 35].

Theorem 3.2. *Let* (1) *be a monotone system so that the origin is asymptotically stable. Suppose that the compact set* $X \subset \mathbb{R}^n_+$ *is positively invariant and that* $\epsilon\mathbf{1} + \sup X$ *is contained in the region of attraction of the origin for some* $\epsilon > 0$. *Then there exist strictly increasing, positive definite, and lower-semicontinuous functions* $V_k : \mathbb{R}_+ \to \mathbb{R}_+$, *for* $k = 1, \ldots, n$ *such that* $V(x) = \max\{V_1(x_1), \ldots, V_n(x_n)\}$ *is a strict Lyapunov function on* X *which additionally satisfies condition 2 of Theorem 2.2. In particular, one has*

$$V^*(x) \leq \dot{V}(x) \leq -V(x).$$

Remark 2. If a compact set X is not positively invariant to begin with, then one can consider instead the positively invariant set

$$Y := \overline{\bigcup_{t \geq 0} \varphi(t, X)}.$$

Proof. Define $\overline{x} = \sup X + \epsilon\mathbf{1}$ for a sufficiently small $\epsilon > 0$ such that $\overline{x}$ is in the region of attraction (which is an open set). Then, due to monotonicity of the system we have for all $x \in X$ that

$$0 \leq \max_k \varphi_k(t, x) \leq \max_k \varphi_k(t, \overline{x}) \longrightarrow 0 \qquad \text{as } t \to \infty$$

where $\varphi_k(t, \overline{x})$ denotes the kth component of $\varphi(t, \overline{x})$. For $x \in X$ define

$$T_k(x_k) := \max\left\{\tau\colon\ x_k \leq \varphi_k(t, \overline{x}) \text{ for all } t \in [0, \tau]\right\}$$
$$T(x) := \max\left\{\tau\colon\ x \leq \varphi(t, \overline{x}) \text{ for all } t \in [0, \tau]\right\}$$

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in p.8
"Discrete and Continuous Dynamical Systems - Series B (DCDS-B)"

where x_k and $\varphi_k(t,\overline{x})$ denote the kth components of x and $\varphi(t,\overline{x})$. From the definition it is clear that the functions T_k and T are *upper-semicontinuous*, i.e., that

$$\limsup_{y\to x} T(y) \leq T(x)$$

holds for all $x \in X$. Then $T(x) = \min\{T_1(x_1), \dots, T_n(x_n)\}$. It follows from compactness of X and asymptotic stability of $x = 0$ that $T(x)$ is finite for all $x \in X$ with $x \neq 0$. Moreover, for $x \neq 0$ and arbitrary $\epsilon > 0$,

$$\begin{aligned} T(\varphi(\epsilon,x)) &= \max\big\{\tau\colon\ \varphi(\epsilon,x) \leq \varphi(t,\overline{x}) \text{ for all } 0 \leq t \leq \tau\big\} \\ &= \max\big\{\tau\colon\ \varphi(\epsilon,x) \leq \varphi(t,\overline{x}) \text{ for all } \epsilon \leq t \leq \tau\big\} \\ &= \max\big\{\tau\colon\ \varphi(\epsilon,x) \leq \varphi(t+\epsilon,\overline{x}) \text{ for all } 0 \leq t \leq \tau-\epsilon\big\} \\ &\geq \max\big\{\tau\colon\ x \leq \varphi(t,\overline{x}) \text{ for all } 0 \leq t \leq \tau-\epsilon\big\} \\ &= \epsilon + T(x) \end{aligned}$$

The inequality is due to monotonicity of the dynamics. This shows that the map $t \mapsto T(\varphi(t,x))$ is a strictly increasing function of t. Note that the same reasoning applies to the functions T_k as well. It follows that the functions

$$V_k(z) := e^{-T_k(z)}, \qquad\qquad k = 1,\dots,n$$

are strictly increasing in z, satisfy $V_k(0) = 0$, and are strictly decreasing along trajectories. With

$$V(x) := \max\big\{V_1(x_1),\dots,V_n(x_n)\big\} = e^{-T(x)}$$

we obtain a max-separable Lyapunov function, as desired. Note that again by construction the functions V_k and V are lower-semicontinuous. For $x \neq 0$ the Dini derivative $\dot{V}$ is estimated as

$$\begin{aligned} \dot{V}(x) &= \liminf_{h\searrow 0} \frac{V(\varphi(h,x)) - V(x)}{h} \\ &= \liminf_{h\searrow 0} \frac{e^{-T(\varphi(h,x))} - e^{-T(x)}}{h} \\ &\leq \liminf_{h\searrow 0} \frac{e^{-(h+T(x))} - e^{-T(x)}}{h} \\ &= \liminf_{h\searrow 0} e^{-T(x)} \frac{e^{-h}-1}{h} = -e^{-T(x)} = -V(x). \end{aligned}$$

The remainder of the claim follows by Remark 1. □

Corollary 2. *If $\overline{x} \gg 0$ is such that $\varphi(t,\overline{x}) \to 0$ as $t \to \infty$, then for any $\overline{y}$ with $0 \ll \overline{y} \ll \overline{x}$ there exists a max-separable Lyapunov function defined on the compact and positively invariant set $X := \bigcup_{t\geq 0}\{x \in \mathbb{R}^n_+ \colon x \leq \varphi(t,\overline{y})\}$.*

The proof follows along the lines of the proof of Theorem 3.2 with a minor modification: For points $x \in X$ satisfying $0 \ll x \not\leq \overline{x}$, we have to use $T(x) := \inf\{\tau \geq 0\colon x \not\ll \varphi(t,\overline{x})\ \forall t \geq \tau\} = \max_k T_k(x)$ with $T_k(x) := \inf\{\tau \geq 0\colon x_k \geq \varphi_k(t,\overline{x})\ \forall t \geq \tau\}$. These definitions coincide with the original ones for $0 \ll x \ll \overline{x}$.

The following example demonstrates the geometric reasoning behind the previous theorem and corollary.

Example 1. Starting from the compact order interval $[0, x^0]$ a compact and positively invariant set $X = \bigcup_{t\geq 0}\big\{x \in \mathbb{R}^2_+ \colon x \leq \varphi(t,x^0)\big\}$ is computed numerically, see Figure 1. For $x^0 = (4, 1.5)$ and $\epsilon = 1$ we obtain $\overline{x} = (5.17, 2.5)$. The trajectory $\varphi(t,\overline{x})$ of (15)–(16) for $t \in \mathbb{R}_+$ is then used to compute the Lyapunov function V as in the proof of Theorem 3.2. The reference trajectory $\varphi(t,\overline{x})$, the set Ω, as well

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in
"Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.9

as sublevel sets $L(l)$ for several values of $l > 0$ (dashed lines) are shown in Figure 1 for

$$g(x_1) = \frac{16x_1}{25(1+x_1)}. \tag{19}$$

The sublevel sets $L(l)$, illustrated in Figure 1 as red dashed rectangles, are domains of attraction, because these sets are bounded.

However, when V is considered as a function defined on all of $\mathbb{R}^2_+$, or at least on the strip $\{x \in \mathbb{R}^2_+ \colon\ x_2 \le \overline{x}_2\}$, then for large $l > 0$ the sublevel sets are unbounded in the x_1-direction and the dashed lines denoting their boundary are horizontal (dash-dotted in the figure). The unboundedness of the x_1-direction occurs when $l > 0$ is chosen such that the sublevel set $L(l)$ exceeds $x_2 = 1$.

It is therefore only possible to extract finely grained information about the attraction rate of the origin in the neighborhood where the sublevel sets of V are bounded.

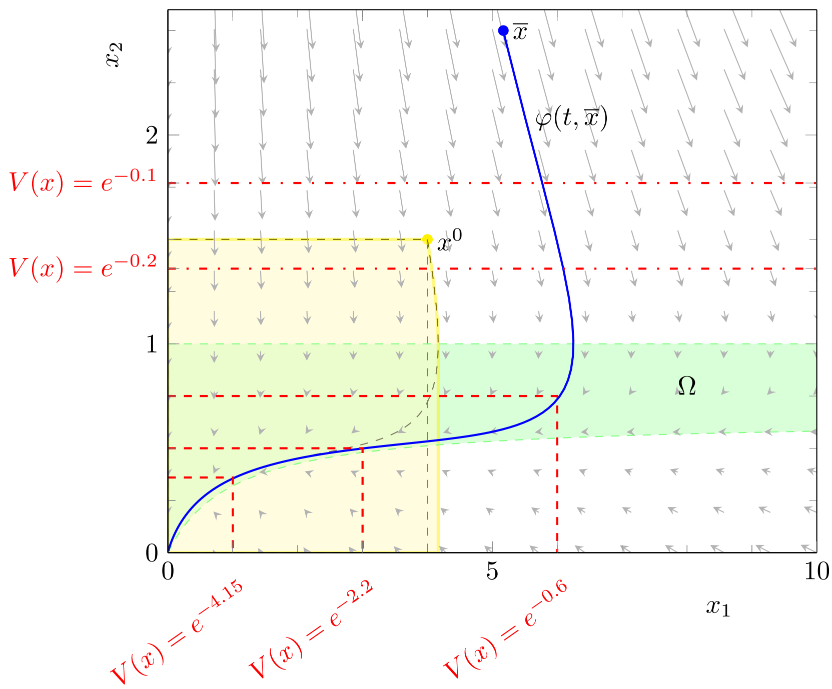

FIGURE 1. Sublevel sets of a max-separable Lyapunov function constructed by Theorem 3.2 for (15)–(16) with g given by (19) in Example 1.

The following example shows that in principle discontinuity of V in the construction of Theorem 3.2 can occur. While in general discontinuous Lyapunov functions are undesirable for most applications, they have their use, e.g., in establishing weak attraction, cf. [5, Chapter IX]. For the monotone systems considered here, they are in fact enough to establish asymptotic stability, cf. Theorem 2.2.

The idea is pictured in Figure 2: If x moves downwards along the vertical line at $x_1 = 2$, the value of $V(x)$ takes a jump when x_2 crosses the value $\tilde{x}_2$ for which the x_2-component of $\varphi(t, \overline{x})$ remains constant for some open time interval (t_1, t_2). However, such a phenomenon cannot occur for the class of monotone systems on

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in "Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.10

$\mathbb{R}^2_+$ considered here: The trajectory depicted in the figure evolves initially in the set Ω, as both components are strictly decreasing with time (e.g., at time $\tilde{t}_1$ in Figure 2), then at time $\tilde{t}_2$ we must have $\frac{d}{dt}\varphi_2(\tilde{t}_2, \overline{x}) = 0$, so this point cannot be in Ω, while at time $\tilde{t}_3$ the trajectory evolves again in Ω. However, by Lemma 2.1 the set Ω is positively invariant, contradicting the existence of a trajectory as shown in Figure 2. Furthermore, it is known that both components of $\varphi(t, \overline{x})$ must eventually be decreasing, cf. [11].

Therefore, the following constructive example is given for a monotone system evolving on $\mathbb{R}^3_+$. In this construction the reference solution $\varphi(t) := \varphi(t, \overline{x})$ evolves

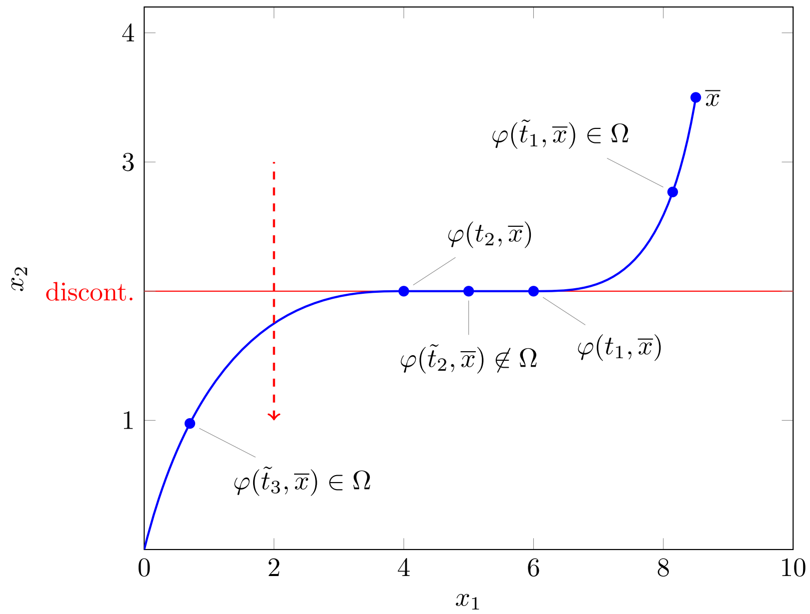

Figure 2. Sketch of the possibility of a discontinuity of the Lyapunov function constructed in Theorem 3.2. Here the x_2 component of $\varphi(t, \overline{x})$ is constant between times t_1 and $t_2 > t_1$, so that $V(x)$, for x sliding down along the red dashed line, jumps from e^{-t_1} to e^{-t_2} when the thin red line at $x_2 = 2$ is crossed.

for a while in a plane parallel to the x_1-x_2-plane. More precisely, while the x_3-component of $\varphi(t)$ is constant for a short time, the dynamics of the x_1- and x_2-components obey the linear dynamics (20) to guarantee monotonicity of the overall system.

Example 2. Consider the following vector field in $\mathbb{R}^3_+$:

$$f(x_1, x_2, x_3) := \begin{pmatrix} -2x_1 + x_2 \\ x_1 - 2x_2 \\ -\big(\rho(x_1) + \rho(x_2)\big)x_3 \end{pmatrix}$$

where $\rho := \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with the following properties:

1. $\rho(r) = c > 0$ for $r \leq r_1$ and $\rho(r) = 0$ for $r \geq r_2$
2. $\rho(r) \in (0, c)$ for $r_1 \leq r \leq r_2$
3. ρ monotonically decreasing, i.e., $\rho'(r) \leq 0$.

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in "Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.11

The values of $r_1 > 0$ and $r_2 > r_1$ will be fixed appropriately later. A cubic spline interpolation based on the above conditions yields

$$\rho(r) = \begin{cases} 0 & \text{if } r < r_1 \\ -c\frac{2r^3+6rr_1r_2-3r_1r_2^2+r_2^3-3(r_1+r_2)r^2}{r_1^3-3r_1^2r_2+3r_1r_2^2-r_2^3} & \text{if } r \in [r_1, r_2] \\ c & \text{for } r > r_2. \end{cases}$$

as one possible choice. Then one has

$$Jf(x,y,z) := \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ -\rho'(x)z & -\rho'(y)z & -\rho(x)-\rho(y) \end{pmatrix}$$

and thus $\dot{x} = f(x)$ is monotone (because Jf has nonnegative off-diagonal entries, cf. the discussion below (8)). Moreover, $\mathbb{R}^3_+$ is obviously positively invariant and the origin is globally asymptotically stable. The last assertion follows easily from the global asymptotic stability of the linear subsystem

$$\begin{aligned} \dot{x}_1 &= -2x_1 + x_2 \\ \dot{x}_2 &= x_1 - 2x_2. \end{aligned} \tag{20}$$

Now consider the compact set $X := K \times [0,1] \subset \mathbb{R}^3_+$, where $K \subset \mathbb{R}^2$ denotes the compact subset bounded by the x-axis and the solution

$$\psi(t) = \frac{5}{2}\left(e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + e^{-3t}\begin{pmatrix}1\\-1\end{pmatrix}\right)$$

of (20) which starts at $(5,0)$. Here, the value 5 does not play a special role; many other choices would do the same job. Next, let us denote the maximum of the x_2-component of $\psi(t)$ by y_*. For our particular initial condition we find $y_* = \frac{5}{3\sqrt{3}} < 1$. Then, according to Theorem 3.2, let us choose $\bar{x} := (5, y_*, 1) + (1,1,1) = (6, y_*+1, 2)$ as starting point for the *reference solution* $\varphi(t)$. Since $\dot{x}_2 = x_1 - 2x_2 > 0$ at $(6, y_*+1)$ the maximum of the x_2-component of $\varphi(t)$ will be greater than y_*+1. Let us denote its maximum by $y^* > 0$. In our particular case we find $y^* = \frac{\sqrt{5}(21\sqrt{3}+5)^{\frac{3}{2}}}{135\sqrt{3\sqrt{3}-1}} \approx 2.15$. Now choose $r_2 > r_1 > 0$ such that

$$r_1 < y_* + 1 < r_2 < y^*.$$

If finally $c > 0$ is sufficiently large, the x_3-component of $\varphi(t)$ will initially decrease fast enough so that one can guarantee that the x_3-component of $\varphi(t)$ is less than 1 when $\varphi(t)$ enters the region $y \geq r_2$, cf. Figure 3. Finally, along the x_3-axis (for some $(x_3 < 1)$ a discontinuity phenomenon for T occurs, which results from the discontinuity of T_3 as shown in Figure 4.

We stress that the discontinuity phenomenon is a non-local property, as for sufficiently regular f in an arbitrarily small neighborhood one may always assume that there is a Lipschitz-continuous max-separable Lyapunov function, as we have seen in Theorem 3.1.

3.1.2. *Global max-separable Lyapunov functions.* In general, the reasoning of Theorem 3.2 does not provide a global Lyapunov function for an arbitrary monotone system with globally asymptotically stable origin, as we will see in the examples of Section 4.1. However, the following result holds.

Corollary 3. *Let* (1) *be a monotone system so that the origin is globally asymptotically stable. Suppose that there is a trajectory* $\overline{\varphi}(t) \in \mathbb{R}^n_+$ *such that*

- $\overline{\varphi}(t)$ *is defined for all forward and backward times;*
- $\lim_{t\to\infty} \overline{\varphi}(t) = 0$ *and* $\lim_{t\to-\infty} \overline{\varphi}_k(t) = \infty$ *for all* k.

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in "Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.12

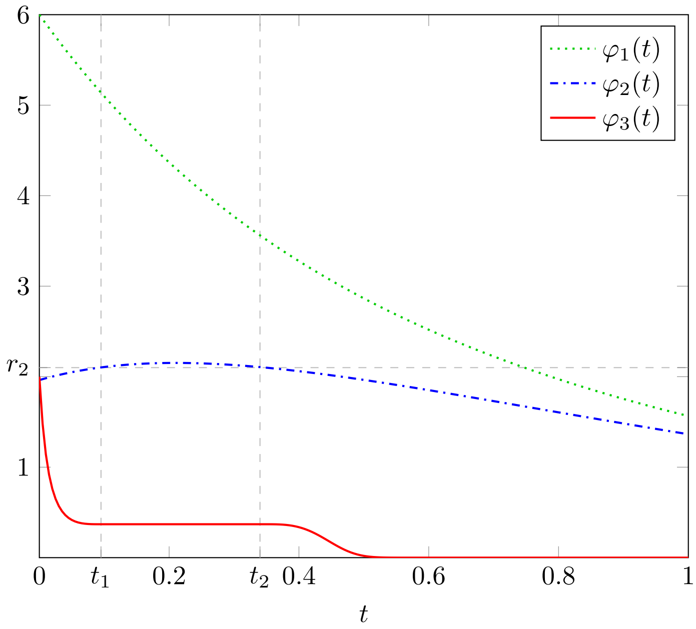

FIGURE 3. The components of the reference trajectory $\varphi(t) = \varphi(t, \overline{x})$ as constructed in Example 2. The trajectories shown here correspond to the numerical values $\overline{x} = \big(6, 5/(3\sqrt{3}), 2\big)$, $r_1 = 1$, $r_2 = 2.1$, and $c = 1500$. Clearly, while $\varphi_2(t) > r_2$ holds, i.e., between t_1 and t_2, the third component $\varphi_3(t)$ is constant.

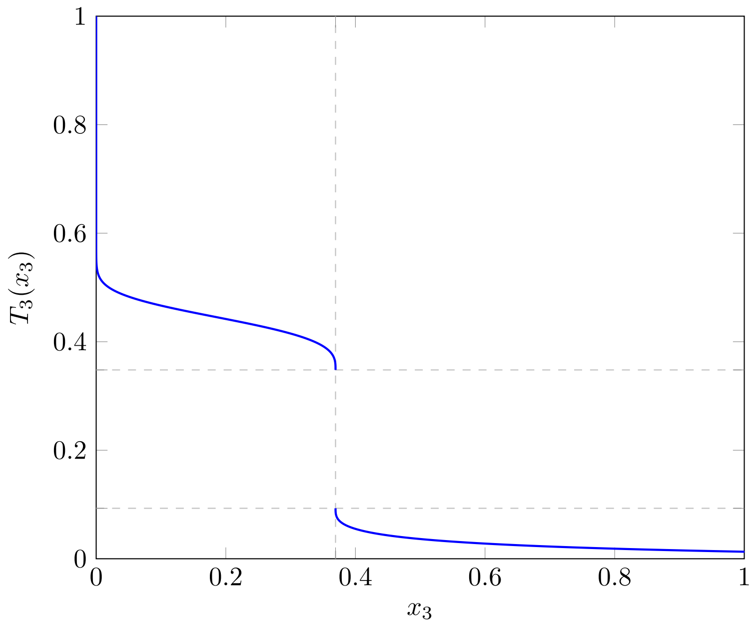

FIGURE 4. The discontinuity of $T_3(x_3) = \max\big\{\tau\colon x_3 \leq \varphi_3(t, \overline{x}) \text{ for all } t \in [0, \tau]\big\}$ as constructed in Example 2. The resulting Lyapunov function V constructed as per Theorem 3.2 is discontinuous.

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in p.13
"Discrete and Continuous Dynamical Systems - Series B (DCDS-B)"

Then there exists a max-separable Lyapunov function.

Proof. The proof is essentially the same as the construction given in the proof of Theorem 3.2. First we let

$$T_k(x_k) := \max\{\tau\colon\ x_k \le \overline{\varphi}_k(t) \text{ for all } t \in (-\infty, \tau]\}$$
$$T(x) := \max\{\tau\colon\ x \le \overline{\varphi}(t) \text{ for all } t \in (-\infty, \tau]\}$$

for $k = 1, \dots, n$. Again $T(x) = \min_k T_k(x_k)$ and we define

$$V(x) := e^{-T(x)} = \max_k e^{-T_k(x_k)} =: \max_k V_k(x_k).$$

Observe that $V(x) \to \infty$ as $\|x\| \to \infty$. The remainder of the proof is the same as for Theorem 3.2. □

3.1.3. *Limitations for planar systems.* In this subsection we consider system (1) with $f\colon \mathbb{R}^2_+ \to \mathbb{R}^2$. Max-separable Lyapunov functions possess the following property when their derivative is negative definite on $\mathbb{R}^2_+$.

Theorem 3.3. *Suppose that there exists a non-decreasing continuous function $\eta\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that the implication*

$$\eta(x_1) \le x_2 \ \Rightarrow\ f_1(x) \ge 0 \tag{21}$$

holds for all $x \in \mathbb{R}^2_+$. If there exist differentiable functions $\rho_1, \rho_2 \in \mathcal{K}$ such that $V\colon \mathbb{R}^2_+ \to \mathbb{R}_+$,

$$V(x) = \max\{\rho_1(x_1), \rho_2(x_2)\} \tag{22}$$

satisfies at all points $x > 0$ with $\rho_1(x_1) \neq \rho_2(x_2)$,

$$\nabla V(x) f(x) < 0, \tag{23}$$

then

$$\lim_{s\to\infty} \rho_1(s) \le \lim_{s\to\infty} \rho_2 \circ \eta(s). \tag{24}$$

In essence this theorem is a non-existence result for global max-separable Lyapunov functions, cf. Figure 9: For certain right hand sides f satisfying (21), the Lyapunov function V cannot be radially unbounded, because due to (24) the function ρ_1 is bounded by the composition of a class $\mathcal{K}$ function with a bounded function and thus must itself be bounded.

Proof. Let $p = (x_1, x_2)^T \in \mathbb{R}^2_+ \setminus \{0\}$ be such that

$$\eta(x_1) \le x_2. \tag{25}$$

Then properties (21), (23) and $\rho_1, \rho_2 \in \mathcal{K}$ imply $\rho_1(x_1) < \rho_2(x_2)$. Since the above property holds for any $p = (x_1, x_2)^T \in \mathbb{R}^2_+ \setminus \{0\}$ satisfying (25), we have

$$\rho_1(s) \le \rho_2 \circ \eta(s), \quad \forall s \in \mathbb{R}_+.$$

Taking the limit of both sides in the above inequality for s tending to ∞, we arrive at (24). □

The preceding result has two immediate consequences.

Corollary 4. *Under the assumptions of Theorem 3.3, if*

$$\lim_{s\to\infty} \eta(s) < \infty$$

holds then every sublevel set $L(l)$ that contains a point $x = (x_1, x_2) \in \mathbb{R}^2_+$ with $x_2 \ge \lim_{s\to\infty} \eta(s)$ is unbounded.

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in p.14
"Discrete and Continuous Dynamical Systems - Series B (DCDS-B)"

Proof. By virtue of Theorem 3.3, due to (22) and (24), the requirement

$$L(l) \cap \left\{x \in \mathbb{R}^2_+ \colon x_2 \geq \lim_{s\to\infty} \eta(s)\right\} \neq \emptyset \tag{26}$$

yields $\lim_{s\to\infty} \rho_1(s) \leq l$ and

$$V(x) \geq l \;\Rightarrow\; V(x) = \rho_2(x_2).$$

Thus, property (26) implies that the sublevel set $L(l)$ contains points (x_1, x_2) with arbitrary $x_1 \in \mathbb{R}_+$, i.e., $L(l)$ is unbounded. □

Corollary 5. *Under the assumptions of Theorem 3.3, if a sublevel set $L(l)$ of V given by* (22) *is bounded then it holds that*

$$\rho_2^{-1}(l) < \eta \circ \rho_1^{-1}(l) < \infty. \tag{27}$$

Here it is actually sufficient to require $\nabla V(x) f(x) < 0$ only for points $x \in L(l)$, as opposed to (23), where this is assumed also outside $L(l)$.

Proof. The boundedness of $L(l)$ implies $l < \lim_{s\to\infty} \rho_i(s)$ for $i = 1, 2$. From $\rho_i \in \mathcal{K}$ it follows that $\rho_i^{-1}(l)$ is well-defined and satisfies $0 < \rho_i^{-1}(l) < \infty$ for $i = 1, 2$. Let $p = \left(\rho_1^{-1}(l), \rho_2^{-1}(l)\right)$. By definition, $p \in L(l) \subseteq \mathbb{R}^2_+$ and $V(p) = \rho_1(p_1) = \rho_2(p_2) = l$. Property (23) together with (21) yields $\eta(\rho_1^{-1}(l)) > \rho_2^{-1}(l)$. Hence, property (27) follows from the monotonicity properties of the function η. □

The unboundedness of the sublevel sets established in Corollary 5 implies that the max-separable Lyapunov function (22) cannot ensure the boundedness of solutions to (1) for all $x(0) \in \mathbb{R}^2_+$ if there exists a non-decreasing continuous function $\eta \colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (21) and $\lim_{s\to\infty} \eta(s) < \infty$.

Note that the existence of $\eta \colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (21) and $\lim_{s\to\infty} \eta(s) < \infty$ in Corollary 5 rules out the existence of a trajectory $\overline{\varphi}$ as in Corollary 3 that is unbounded in all components in backward time.

Remark 3. For monotone systems, there is a way to estimate domains of attraction without the use of sublevel sets of Lyapunov functions at all [34, 35, 31].

The reasoning is as follows: If there exists $x_0 \in \mathbb{R}^n_+$ such that the solution $\varphi(t, x_0)$ of (1) satisfies $\lim_{t\to\infty} \varphi(t, x_0) = 0$, then from the monotonicity of (1) it follows that the set

$$B(x_0) \coloneqq \{x \in \mathbb{R}^n_+ \colon \exists t \in \mathbb{R}_+ \text{ such that } x \leq \varphi(t, x_0)\}$$

is a domain of attraction and positively invariant. Note that $B(x_0)$ does not have to be a sublevel set of a max-separable Lyapunov function.

3.2. Sum-separable Lyapunov functions.

3.2.1. *Local sum-separable Lyapunov functions.* We start again with a local result.

Theorem 3.4. *Assume that f is continuously differentiable at the origin and that $Df(0)$ is Hurwitz. Then there exists a sum-separable Lyapunov function in a neighborhood of the origin.*

Proof. Much of this proof is the same as for Theorem 3.1, so we only state the relevant modifications. Denote again by A the Jacobian of f at the origin. If A is irreducible then by the Perron-Frobenius Theorem A has a positive left eigenvector $\lambda \gg 0$ such that $\lambda^T A = aA$ with $a < 0$ the spectral abscissa of A.

The function $V(x) \coloneqq \lambda^T x$ is positive definite on $\mathbb{R}^n_+$, because all components of λ are positive. Along trajectories of the linearized system $\dot{x} = Ax$ we compute $\frac{d}{dt} V(x(t)) = \lambda^T A x = a\lambda^T x = aV(x) < 0$ whenever $x > 0$. The case when A is reducible can again be handled by considering instead $\tilde{A} = A + \epsilon \mathbf{1}$ for a sufficiently small $\epsilon > 0$. The remaining arguments are the same in the proof of Theorem 3.1. □

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in "Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.15

3.2.2. *Global sum-separable Lyapunov functions.* It is not yet known whether sum-separable Lyapunov functions can be generated directly from a single trajectory. However, it is possible to analytically construct sum-separable Lyapunov functions based on information little more than the positively invariant set Ω. In this section we restrict our attention again to the planar case.

We consider the monotone system

$$\dot{x} = f(x) \coloneqq \begin{pmatrix} -\alpha_1(x_1) + \sigma_1(x_2) \\ -\alpha_2(x_2) + \sigma_2(x_1) \end{pmatrix} \tag{28}$$

with $\alpha_i \in \mathcal{K}$ and $\sigma_i \in \mathcal{K}$ for $i = 1, 2$. The monotonicity of (28) is clear since (8) holds.

Remark 4. System (28) can be interpreted as a feedback interconnection of two systems, $\dot{x}_1 = f_1(x_1, x_2)$ and $\dot{x}_2 = f_2(x_1, x_2)$. Both of these systems are integral input-to-state stable with respect to the state of the other system. Moreover, system i is ISS with respect to the state of the other system [42], if and only if $\lim_{s\to\infty} \sigma_i(s) \le \lim_{s\to\infty} \alpha_i(s)$.

As demonstrated in [1], unless $\lim_{s\to\infty} \sigma_i(s) \le \lim_{s\to\infty} \alpha_i(s)$ holds for both $i = 1, 2$, the existence of Ω dividing $(0, \infty)^2$ into two disjoint sets is not sufficient for guaranteeing global asymptotic stability of (28)[2]. For ensuring the asymptotic stability in the global sense, it is sufficient that the "width" of Ω does not shrink to zero in the radial direction. In fact, we can verify the following by making use of the argument in [23], where $c > 1$ prevents Ω from shrinking to zero.

Theorem 3.5. *Suppose that there exists $c > 1$ such that*

$$\alpha_1^{\ominus} \circ c\sigma_1 \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \le s, \tag{29}$$

holds for all $s \in \mathbb{R}_+$. Choose $\psi \in \mathbb{R}_+$ such that

$$\begin{aligned} \psi &= 0 \ \text{if } c > 2 \\ \psi^{-\frac{\psi}{\psi+1}} &< \frac{c}{\psi+1} \le 1 \ \text{otherwise.} \end{aligned} \tag{30}$$

Then the continuously differentiable function $V \colon \mathbb{R}_+^2 \to \mathbb{R}_+$ defined by

$$V(x) = \rho_1(x_1) + \rho_2(x_2) \tag{31}$$

$$\rho_i(s) = \int_0^s \lambda_i(\tau)d\tau, \quad i = 1, 2 \tag{32}$$

$$\lambda_i(s) = \alpha_i(s)^{\psi}\sigma_{3-i}(s)^{\psi+1}, \ i = 1, 2 \tag{33}$$

is a Lyapunov function.

Remark 5. Condition (29) is called a small-gain condition [1, 21, 20].

Proof. Let $\tau > 1$. For λ_i given in (33), we have

$$\begin{aligned} &\lambda_i(x_i)\{-\alpha_i(x_i) + \sigma_i(x_{3-i})\} \le \\ &\quad -\left(1 - \frac{1}{\tau}\right)\alpha_i(x_i)^{\psi+1}\sigma_{3-i}(x_i)^{\psi+1} \\ &\quad + \tau^{\psi}\sigma_i(x_{3-i})^{\psi+1}[\sigma_{3-i} \circ \alpha_i^{\ominus} \circ \tau\sigma_i(x_{3-i})]^{\psi+1} \end{aligned}$$

for $i = 1, 2$. Define

$$Q_i(x_i) \coloneqq \left(1 - \frac{1}{\tau}\right)\alpha_i(x_i)^{\psi+1} - \tau^{\psi}[\sigma_i \circ \alpha_{3-i}^{\ominus} \circ \tau\sigma_{3-i}(x_i)]^{\psi+1}$$

[2] The existence of Ω dividing $(0, \infty)^2$ into two disjoint sets is sufficient if we are only interested in compact domains of attraction.

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in p.16
"Discrete and Continuous Dynamical Systems - Series B (DCDS-B)"

for $i = 1, 2$. If

$$1 < \tau \leq c \tag{34}$$

holds, by virtue of (29) we have

$$Q_i(x_i) \geq \left(1 - \frac{1}{\tau}\right)\alpha_i(x_i)^{\psi+1} - \tau^{\psi}\left[\frac{1}{c}\alpha_i(x_i)\right]^{\psi+1}.$$

Thus, if

$$\left(\frac{\tau}{c}\right)^{\psi+1} < \tau - 1 \tag{35}$$

is satisfied, there exists $\epsilon > 0$ such that

$$Q_i(x_i) \geq \epsilon[\alpha_i(x_i)]^{\psi+1}.$$

Since we have

$$\sum_{i=1}^{2}\lambda_i(x_i)\{-\alpha_i(x_i) + \sigma_i(x_{3-i})\} = -\sum_{i=1}^{2}\sigma_{3-i}(x_i)^{\psi+1}Q_i(x_i),$$

property (23) is achieved for all $x \in \mathbb{R}_+^2$ if $\psi, \tau \geq 0$ satisfy (34) and (35). Finally, we prove that there exists $\tau > 0$ such that (34) and (35) are satisfied if ψ satisfies (30). First, suppose that $\psi = 0$. Let

$$\tau = \frac{c}{2} + 1.$$

Then $c > 2$ guarantees (34). We also have

$$Z \coloneqq \tau - 1 - \frac{\tau}{c} = \frac{(c+1)(c-2)}{2c}.$$

From $c > 2$ it follows $Z > 0$ which is identical to (35). Next, suppose that $\psi > 0$. Let

$$\tau = c\left(\frac{c}{\psi+1}\right)^{\frac{1}{\psi}}. \tag{36}$$

Then property

$$1 \geq \frac{c}{\psi+1}$$

implied by (30) yields $\tau \leq c$. Since (30) guarantees

$$\left(\frac{c}{\psi+1}\right)^{\frac{\psi+1}{\psi}} > \psi^{-1}, \tag{37}$$

from (36) we obtain

$$\tau = (\psi+1)\left(\frac{c}{\psi+1}\right)^{\frac{1}{\psi}+1} > \frac{\psi+1}{\psi} > 1.$$

Hence, we arrive at (34). Using (36) we have

$$Z \coloneqq \tau - 1 - \left(\frac{\tau}{c}\right)^{\psi+1} = \psi\left(\frac{c}{\psi+1}\right)^{\frac{1}{\psi}+1} - 1.$$

Property (37) yields $Z > 0$ which is identical to (35). This completes the proof. $\square$

By construction V is radially unbounded, so the sublevel sets $L(l)$ defined in (13) are positively invariant. Clearly, the sum-separable Lyapunov function V establishes global asymptotic stability of the origin for system (28).

We stress that there always exists $\psi \geq 0$ satisfying (30). In fact, $c \leq \psi + 1$ is met for a sufficiently large $\psi \geq 0$, and we have

$$\lim_{\psi \to \infty} (\psi + 1)\psi^{-\frac{\psi}{\psi+1}} = 1.$$

Due to $c > 1$ and continuity, the requirement (30) holds whenever $\psi \geq 0$ is sufficiently large. The smallest ψ satisfying (30) for given $c > 1$ is shown in Figure 7.

Example 3. Consider the monotone system

$$\dot{v}_1 = -\hat{b}(v_1) + v_2 \tag{38}$$

$$\dot{v}_2 = -2v_2 - v_2^2 + 2\hat{g}(v_1) + \hat{g}(v_1)^2 \tag{39}$$

defined for $v = (v_1, v_2) \in \mathbb{R}^2_+$, where $a \geq 1$ and

$$b(s) = \frac{as}{1+as}, \quad \hat{b}(s) = b\left(\frac{e^s - 1}{a}\right), \quad s \in \mathbb{R}_+$$

$$\hat{g}(s) = g\left(\frac{e^s - 1}{a}\right), \quad s \in \mathbb{R}_+.$$

Applying the diffeomorphisms $v_1 = \log(1 + ax_1)$ and $v_2 = x_2$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ to (15) and (16) gives

$$\dot{v}_1 = -\frac{ax_1}{1+ax_1} + \frac{ax_1}{1+ax_1}x_2 \leq -\hat{b}(v_1) + v_2$$

and (39). Thus, due to the standard argument of the comparison principle (e.g. [26, 27]), a domain of attraction of (38)–(39) (in x-coordinates) is a domain of attraction of the original system (15)–(16).

In order to apply Theorem 3.5 for constructing a Lyapunov function $V(v)$, suppose that g is given as (19) and let $a = 1$. Then the functions

$$\alpha_1(s) = \hat{b}(s), \ \sigma_1(s) = s$$

$$\alpha_2(s) = 2s + s^2, \ \sigma_2(s) = 2\hat{g}(s) + \hat{g}(s)^2$$

satisfy (29) with $c \in (1, 5/4]$. Equations (33) and (30) with $c = 5/4$ lead to

$$\lambda_1(s) = \hat{b}(s)^{17}(2\hat{g}(s) + \hat{g}(s)^2)^{18} \text{ and} \tag{40}$$

$$\lambda_2(s) = (2s + s^2)^{17} s^{18}. \tag{41}$$

From (31) a Lyapunov function V is obtained as

$$V(x) = \rho_1(v_1) + \rho_2(v_2) = \rho_1(\log(1 + x_1)) + \rho_2(x_2) \tag{42}$$

with (32). Sublevel sets $L(l)$ of (42) are plotted for several $l > 0$ in Figure 5, where

$$\Omega_v = \{x \in \mathbb{R}^2_+ : \ \alpha_i(v_i) > \sigma_i(v_{3-i}), \ i = 1, 2\}.$$

All sublevel sets are compact and thus domains of attractions, although some parts of large x_1 exceed the frame of Figure 5. In fact, both ρ_1 and ρ_2 generated from (40) and (41) via (32) are radially unbounded, and so is V. This implies that for an arbitrarily large $x \in \mathbb{R}^2_+$, there always exists $l > 0$ such that $x \in L(l)$. Therefore, the Lyapunov function (42) establishes global asymptotic stability of $x = 0$.

The sublevel sets obtained from the sum-separable Lyapunov function constructed in the previous example still show almost 'sharp' upper right corners. This shape is due to the diameter of the the set Ω, as the following modification of Example 3 shows:

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in "Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.18

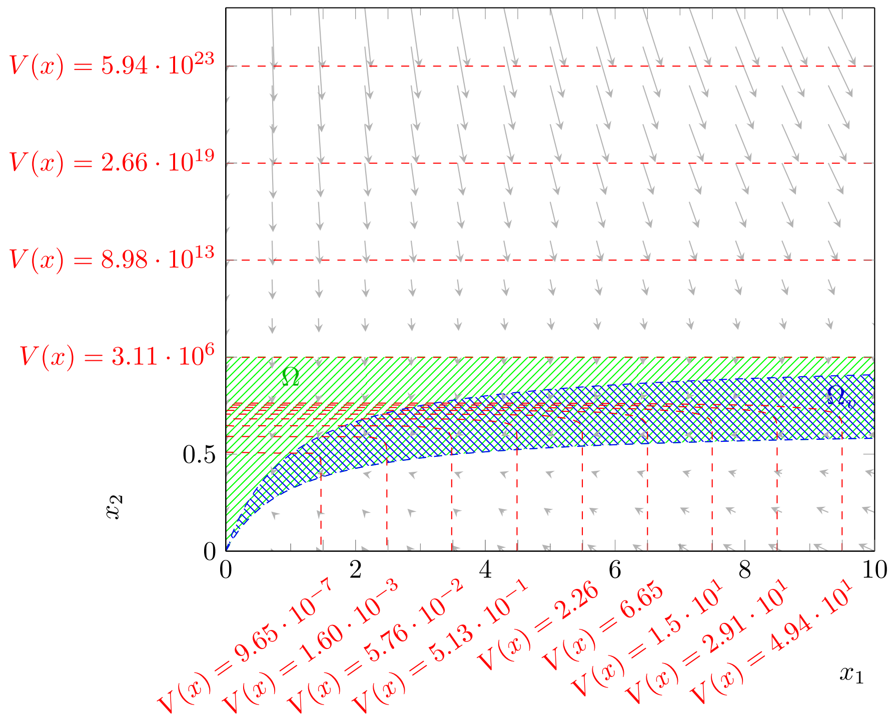

FIGURE 5. Sublevel sets of a sum-separable Lyapunov function for (15)–(16) and (19) in Example 3 via Theorem 3.5.

Example 4. The sublevel sets become more rounded and well-balanced in both x_1 and x_2 directions if Ω (or Ω_v) is wider. To see this, we replace (19) by

$$g(x_1) = \frac{6x_1}{25(1+x_1)}. \tag{43}$$

Property (29) is satisfied with $c = \sqrt{25/6}$. Then from $\psi = 0$ in (33) satisfying (30) it follows that

$$\lambda_1(s) = 2\hat{g}(s) + \hat{g}(s)^2 \text{ and} \tag{44}$$

$$\lambda_2(s) = s. \tag{45}$$

For (43) sublevel sets $L(l)$ are plotted in Figure 6. It may illustrate better than Figure 5 that there always exists $l > 0$ such that $L(l)$ is large enough to contain any given $x \in \mathbb{R}^2_+$.

Note that the choice of the pair (ρ_1, ρ_2) in (31) establishing global asymptotic stability of the origin for system (28) is not unique.

Corollary 6. *Suppose that there exist $c_i > 0$, $i = 1, 2$, and $k > 0$ such that*

$$\sigma_2(s)^k \le c_1\alpha_1(s), \quad \forall s \in \mathbb{R}_+ \tag{46}$$

$$c_2\sigma_1(s) \le \alpha_2(s)^k, \quad \forall s \in \mathbb{R}_+ \tag{47}$$

$$c_1 < c_2. \tag{48}$$

Then there exists a constant $c > 1$ satisfying (29).

Remark 6. The existence of $c_1, c_2, k > 0$ satisfying (46)–(48) allows us to replace (33) by another formula for constructing a Lyapunov function via (31)–(32). In

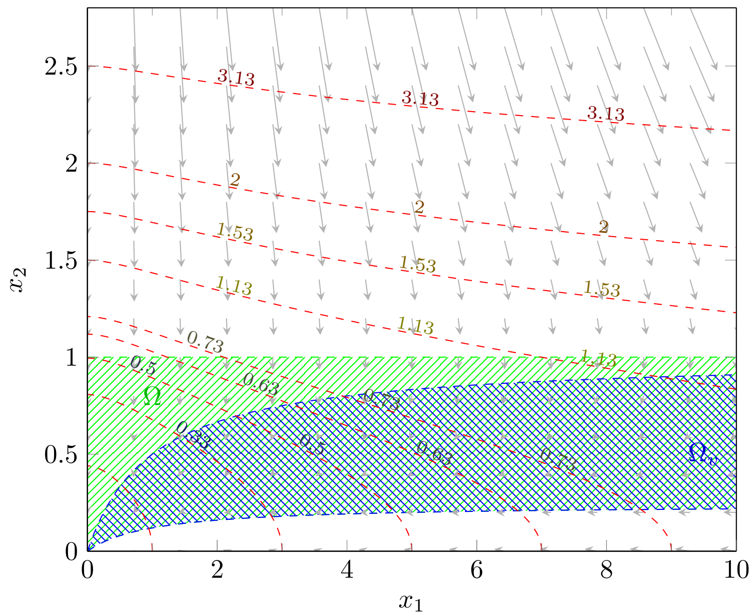

FIGURE 6. Sublevel sets of a sum-separable Lyapunov function for (15)–(16) and (43) in Example 4 via Theorem 3.5.

fact, according to [21], in the case of $k \geq 1$, we can verify that the pair

$$\lambda_1 = c_1 \left(\frac{c_2}{c_1}\right)^{\frac{k+1}{k+2}}, \quad \lambda_2(s) = k\alpha_2(s)^{k-1} \tag{49}$$

satisfies (23) for all $x \in \mathbb{R}^2_+$. For $k < 1$, the above pair is replaced by

$$\lambda_1(s) = \frac{1}{k}\alpha_1(s)^{\frac{1-k}{k}}, \quad \lambda_2 = c_1^{-\frac{1}{k}} \left(\frac{c_1}{c_2}\right)^{\frac{1}{1+2k}} \tag{50}$$

Again, the function V obtained with (49) and (50) is positive definite and radially unbounded.

The case that $\sigma_i \equiv 0$ for $i = 1$ or $i = 2$ in (28) can be dealt with by a robustness argument, replacing the σ_i by a very "small" function. A more explicit case is considered next, where only σ_2 is zero.

$$\dot{x} = f(x) := \begin{pmatrix} -\alpha_1(x_1) + \sigma_1(x_2) \\ -\alpha_2(x_2) \end{pmatrix} \tag{51}$$

with α_1, α_2 positive definite and $\sigma_1 \in \mathcal{K}$.

This scenario can be interpreted as a series connection of two systems. In fact, this very monotone system is a prototype that often arises in the stability analysis of coupled systems via comparison methods.

If α_1 and α_2 are only assumed to be positive definite then depending on the function σ_1 the origin is not always globally asymptotically stable for this type of system. Hence, additional assumptions have to be imposed to allow for the construction of a sum-separable Lyapunov function. The following result follows from a result in [19] which goes beyond Theorem 3.5.

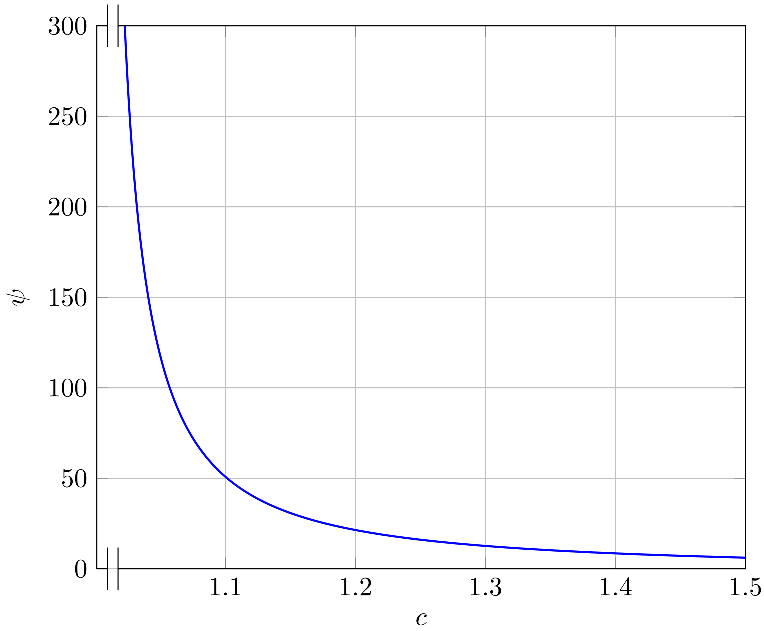

FIGURE 7. Smallest exponent ψ in a sum-separable Lyapunov function given by Theorem 3.5.

Corollary 7 ([19])**.** *Let α_1 and α_2 in (28) be positive definite. Assume there exists $k \geq 1$ such that*

$$\int_1^\infty \alpha_1(s)^{k-1} ds = \infty \text{ and}$$
$$\int_0^1 \frac{\sigma_1(s)^k}{\alpha_2(s)} ds < \infty$$

hold. Then $V : \mathbb{R}^2_+ \to \mathbb{R}_+$ defined by $V(x) = \rho_1(x_1) + \rho_2(x_2)$ with $\rho_i(s) := \int_0^s \lambda_i(\tau)d\tau$ and functions λ_i given by

$$\lambda_1(s) := \frac{1}{2}\alpha_1(s)^{k-1} \text{ and} \tag{52}$$

$$\lambda_2(s) := \begin{cases} \frac{\sigma_1(s)^k}{\alpha_2(s)} & \text{if } s \in [0,1) \\ \max\limits_{w \in [1,s]} \frac{\sigma_1(w)^k}{\alpha_2(w)} & \text{if } s > 1 \end{cases} \tag{53}$$

is a differentiable Lyapunov function.

3.2.3. *Discussion.* Although sum-separable Lyapunov are better than max-separable ones in being able to yield domains of attraction of unlimited size theoretically, the max-separable Lyapunov functions still have some advantages.

In the examples we have seen that the max-separable constructions do not require any pre-process of computing α_i and σ_i from the original system equation. In addition, the functions ρ_i for the max-separable Lyapunov function (22) are independent of α_i and σ_i as long as the sign of $f_i(x)$ remains unchanged. For instance, the max-separable Lyapunov function V obtained above for the system (15)–(16)

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in "Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.21

also has negative time-derivative along solutions of monotone systems such as

$$\dot{x}_1 = -x_1^5 + x_1^5 x_2 \tag{54}$$

$$\dot{x}_2 = (1 + x_2^2)(-x_2 + g(x_1)). \tag{55}$$

On the other hand, the sum-separable Lyapunov function (42) obtained in Example 3 does not have negative time-derivative along solutions of system (54)–(55).

Another benefit of using max-separable Lyapunov functions is their handiness. The exponent ψ appearing in the sum-separable Lyapunov function can be quite large, as we saw for system (40)–(41), cf. Figure 7. In practice, the large exponent causes serious trouble in controller design based on Lyapunov functions when it results in very high "order" nonlinearities in controllers.

In addition, when x_2 is allowed to be larger than unity, the sublevel sets obtained via the sum-separable Lyapunov function for system (15)–(16) are extremely large in the x_1-direction, even though they are guaranteed to remain bounded.

The necessity of high order nonlinearities for sum-separable Lyapunov functions has not been proven. Nevertheless, except for the special case of (46)–(48), we have not yet found ways to reduce the order of the nonlinearities when $c > 1$ in (29) needs to be close to unity. When compared with the max-separable Lyapunov functions, order reduction is naturally harder since the transformation $\kappa(V(x))$ by a class $\mathcal{K}$ function κ generally destroys sum-separability, while it preserves max-separability.

4. **Counterexamples.** The following two examples demonstrate that compactness of the state-space is indeed crucial for the existence of separable Lyapunov functions. In both cases the origin is globally asymptotically stable and the system evolves in $\mathbb{R}^2_+$.

4.1. **Example of a system with a sum-separable Lyapunov function that does not exhibit a max-separable Lyapunov function.** Consider the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{x_1}{1+x_1} + x_2 \\ -x_2 \end{pmatrix} =: f(x_1, x_2). \tag{56}$$

The right-hand side is locally Lipschitz continuous, satisfies $f(0,0) = 0$, as well as the Kamke-Müller conditions (8). Hence (56) defines a monotone system on $\mathbb{R}^2_+$. Figure 8 shows how the state space is divided into two regions,

$$R_{\texttt{upper}} = \left\{ x \in \mathbb{R}^2_+ : \ x_1 > 0,\, x_2 > \frac{x_1}{1+x_1} \right\}$$

$$R_{\texttt{lower}} = \left\{ x \in \mathbb{R}^2_+ : \ x_1 > 0,\, 0 < x_2 < \frac{x_1}{1+x_1} \right\} = \Omega,$$

separated by the dashed line. In the upper region trajectories increase in the first component, while they decrease in the second component. Eventually, they enter the lower region, where both components decrease ad infinitum towards the origin. The shown trajectory is representative for all trajectories passing through $R_{\texttt{upper}}$. Clearly, none of them is unbounded in both components in backward-time. Hence, no trajectory as in Corollary 3 can be used to dominate all points in $\mathbb{R}^n_+$ and the construction of that corollary fails.

Next we show, that there is no "other" max-separable, locally Lipschitz continuous Lyapunov function satisfying (10) either. By way of contradiction assume that there is a $V(x_1, x_2) = \max\{V_1(x_1), V_2(x_2)\}$. Since $V(x_1, 0) = V_1(x_1)$ and $V(0, x_2) = V_2(x_2)$ we conclude that V_i, $i = 1, 2$, are locally Lipschitz continuous and hence differentiable almost everywhere.

Now assume with loss of generality that V_1 is differentiable at all $n \in \mathbb{N}$ and observe that $V(x_1, 0) = V_1(x_1)$ implies $V_1'(s) = \frac{d}{ds}V_1(s) \geq 0$ wherever the derivative

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in "Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.22

exists. Moreover, there must be some $N \geq 1$ such that for all $n \geq N$ we have $V(n,2) = V_1(n)$. Hence for $n \geq N$, we find

$$\begin{aligned} \dot{V}(n,2) &= V_1'(n) f_1(n,2) \\ &= V_1'(n)\left(2 - \frac{n}{1+n}\right) \geq 0. \end{aligned}$$

The same argument would work along any other horizontal line above R_{lower}, so we can actually show that $\dot{V} \geq 0$ on a set of positive measure. This, however, contradicts the fact that V is supposed to be Lyapunov function. Hence, this system does not admit any max-separable Lyapunov function satisfying (10). In fact, this is consistent with Theorem 3.3.

Now consider the $\mathcal{C}^1$ function $V(x_1, x_2) = x_1 + 2x_2$. On $\mathbb{R}^2_+$ it is positive definite and radially unbounded. The system is globally asymptotically stable. We have $\dot{V} = \dot{x}_1 + 2\dot{x}_2 = -\frac{x_1}{1+x_1} - x_2 < 0$ for all $x_1 > 0$ and $x_2 > 0$. So V must be a Lyapunov function, and very clearly it is sum-separable. This establishes that the origin is globally asymptotically stable.

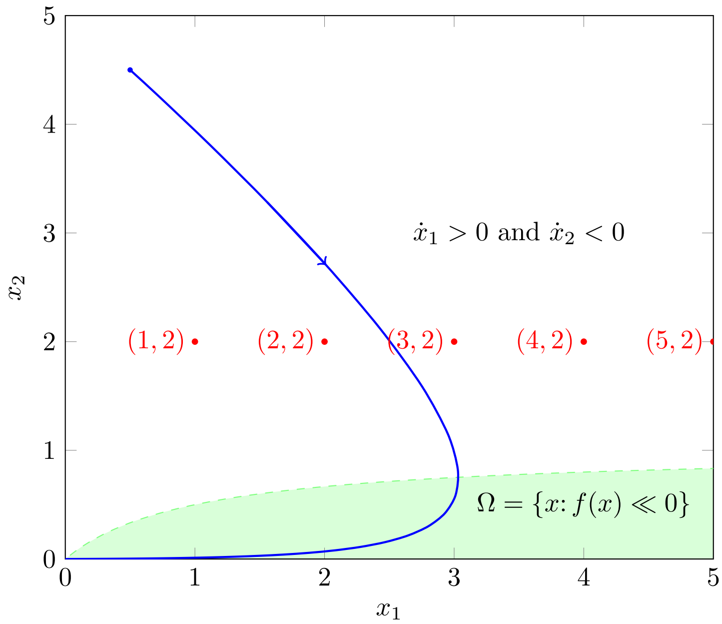

Figure 8. Sign patterns of the right-hand side of system (56) given in Section 4.1. Although the system is globally asymptotically stable, it does not admit a *global* max-separable Lyapunov function. The simple reason is that no trajectory is unbounded in all components in backward-time.

4.2. **Example of a system that does not exhibit a sum-separable nor a max-separable Lyapunov function.** Our second example shows that for non-compact state-space a sum-separable, locally Lipschitz continuous Lyapunov function does not need to exist either.

4.2.1. *Preliminary step.* Consider the following two-dimensional (preliminary) system on $\mathbb{R}_+ \times \mathbb{R}_+$:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \widehat{f}(x_1, x_2) := \begin{pmatrix} \frac{x_2^2}{x_2^2+1} - x_1 \\ x_1 - \frac{2x_2^2}{x_2^2+1} \end{pmatrix} \tag{57}$$

Clearly, if $x_1 > \frac{x_2^2}{x_2^2+1}$ then $\widehat{f}_1(x_1,x_2) < 0$ and if $x_1 < \frac{2x_2^2}{x_2^2+1}$ then $\widehat{f}_2(x_1,x_2) < 0$. Thus, for $\frac{x_2^2}{x_2^2+1} < x_1 < \frac{2x_2^2}{x_2^2+1}$ one has $\widehat{f}_1(x_1,x_2) < 0$ and $\widehat{f}_2(x_1,x_2) < 0$, as depicted in Figure 9.

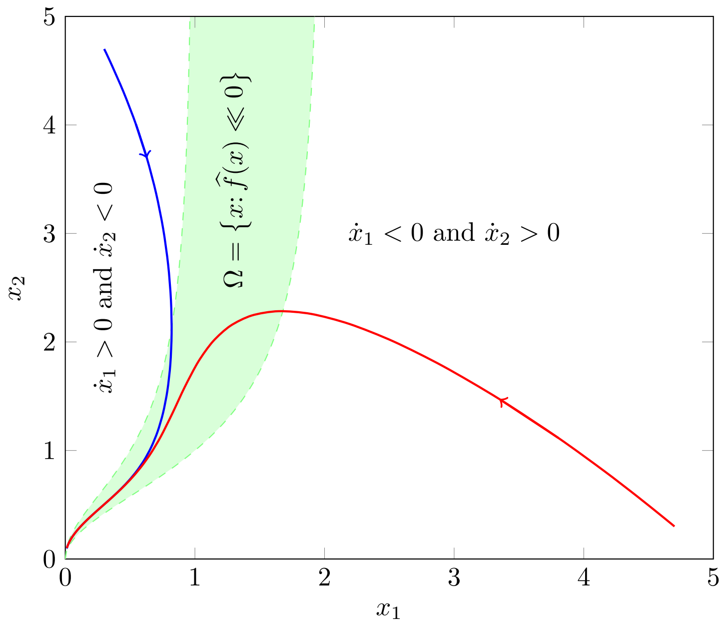

FIGURE 9. Sign patterns of the right-hand side of system (57) given in Section 4.2 and two representative trajectories. Although the system is globally asymptotically stable, it does not admit a *global* sum-separable Lyapunov function.

Now, assume that for (57) there exists a strict global Lyapunov function of the form

$$V(x_1,x_2) = V_1(x_1) + V_2(x_2), \tag{58}$$

i.e., W is supposed to be differentiable (not necessarily continuously differentiable) on $\mathbb{R}^2_+$ and has to satisfy the condition

$$\dot{V}(x_1,x_2) := V_1'(x_1)\widehat{f}_1(x_1,x_2) + V_2'(x_2)\widehat{f}_2(x_1,x_2) < 0 \tag{59}$$

for all $(x_1,x_2) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$, where V_1' and V_2' denote the derivatives of V_1 and V_2, respectively.

4.2.2. *Step 1.* Now, we pass from (57) to the following system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = f(x_1,x_2) := \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix} \tag{60}$$

where f_1 equals $\widehat{f}_1$ and f_2 has the same sign pattern as $\widehat{f}_2$, yet a different limit behaviour. More precisely, there should exist $0 < x_* < 1$ and $x^* > 2$ such that

$$\lim_{x_2\to\infty} f_2(x_*,x_2) = 0 \qquad \text{and} \qquad \lim_{x_2\to\infty} f_2(x^*,x_2) = \infty.$$

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in "Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.24

4.2.3. *Claim.* If there exists a map f_2 with the above properties then (60) does not admit a Lyapunov function of the form (58).

Proof. Assume that (60) has a Lyapunov function of the form (58). This, however, would imply the following (contradictory) limit behaviour for $U'(x_2)$:

(a) On the one hand, the inequality

$$\dot V(x_*, x_2) = \underbrace{V_1'(x_*)}_{>0} f_1(x_*, x_2) + \underbrace{V_2'(x_2)}_{>0} \underbrace{f_2(x_*, x_2)}_{<0} < 0$$

implies $\lim_{x_2\to\infty} V_2'(x_2) = \infty$ because $\lim_{x_2\to\infty} f_1(x_*, x_2) = 1 - x_* > 0$ and $\lim_{x_2\to\infty} f_2(x_*, x_2) = 0$.

(b) On the other hand, the inequality

$$\dot V(x^*, x_2) = \underbrace{V_1'(x^*)}_{>0} \underbrace{f_1(x^*, x_2)}_{<0} + \underbrace{V_2'(x_2)}_{>0} \underbrace{f_2(x^*, x_2)}_{>0} < 0$$

implies $\lim_{x_2\to\infty} V_2'(x_2) = 0$ because $\lim_{x_2\to\infty} f_1(x_*, x_2) = 1 - x^* < 0$ and $\lim_{x_2\to\infty} f_2(x_*, x_2) = \infty$.

□

Thus, once we have shown that such a map f_2 does exist we have also proved that (60) does not admit a Lyapunov function of the form $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$.

4.2.4. *Step 2.* Here, we explicitly "construct" a map f_2 which satisfies the above requirements. Choose continuously differentiable, positive definite functions $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned} \lim_{x_2\to\infty} \alpha(x_2) = 0, \qquad \lim_{x_2\to\infty} \beta(x_2) = \infty, \quad \text{and} \\ \lim_{x_2\to\infty} \alpha(x_2) \mathrm{e}^{\lambda\beta(x_2)} = \infty \end{aligned} \tag{61}$$

for some suitable $\lambda > 0$. Then, define $f_2\colon \mathbb{R}_+^2 \to \mathbb{R}$ as follows

$$f_2(x_1, x_2) := \alpha(x_2)\Big(\mathrm{e}^{\beta(x_2)\left(x_1 - \frac{2x_2^2}{x_2^2-1}\right)} - 1\Big) = \alpha(x_2)\Big(\mathrm{e}^{\beta(x_2)\widehat{f}_2(x_1,x_2)} - 1\Big).$$

Obviously, f_2 has the same sign pattern as $\widehat{f}_2$. Moreover, for $x_* < 2$ and $x^* := 2+\lambda$ one has the following limit behaviour

$$\begin{aligned} \lim_{x_2\to\infty} f_2(x_*, x_2) &= \lim_{x_2\to\infty} -\alpha(x_2) = 0 \quad \text{and} \\ \lim_{x_2\to\infty} f_2(x^*, x_2) &= \lim_{x_2\to\infty} \alpha(x_2)\mathrm{e}^{\beta(x_2)\left(x^*-2\right)} = \infty. \end{aligned}$$

4.2.5. *Step 3.* Finally, we have to choose α and β such that (60) is monotone and asymptotically stable.

Monotonicity According to the discussion below (8), all we have to check is that $\frac{\partial f_1}{\partial x_2}$ and $\frac{\partial f_2}{\partial x_1}$ are non-negative. Indeed, we find

$$\frac{\partial f_1}{\partial x_2}(x_1, x_2) = \frac{2x_2(x_2^2+1) - 2x_2^3}{(x_2^2+1)^2} = \frac{2x_2}{(x_2^2+1)^2} \geq 0$$

and

$$\frac{\partial f_2}{\partial x_1}(x_1, x_2) = \alpha(x_2)\beta(x_2)\mathrm{e}^{\beta(x_2)\left(x_1 - \frac{2x_2^2}{x_2^2-1}\right)} > 0.$$

Thus (60) defines a monotone system on $\mathbb{R}_+ \times \mathbb{R}_+$, whenever α and β are strictly positive.

Asymptotic stability First, the positive invariance of $\mathbb{R}_+ \times \mathbb{R}_+$ under the flow of (60) follows straightforwardly by inspection of the vector field $f = (f_1, f_2)$ on the x_1- and x_2-axis. Moreover, there are obviously no other equilibria in $\mathbb{R}_+ \times \mathbb{R}_+$

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in "Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.25

than $(0,0)$. To prove global asymptotic stability of $(0,0)$, it suffices to show that all solutions of (60) eventually reach the set

$$\Omega = \left\{(x_1,x_2) \in \mathbb{R}^2_+ :\ \tfrac{x_2^2}{x_2^2+1} < x_1 < \tfrac{2x_2^2}{x_2^2+1}\right\}$$

introduced in (9), because Ω is positively invariant under the flow of (60) and admits $W(x_1,x_2) := x_1 + x_2$ as Lyapunov function. Due to the sign pattern of f_1 and f_2, the "attractiveness" of Ω is easily established once one can guarantee that the vector field f is complete (no finite escape time). Therefore, one has to choose α and β in a moderate way, e.g.

$$\alpha(x_2) := \big(\ln(x_2+c)\big)^{-1} \qquad \text{and} \qquad \beta(x_2) := \ln\big(\ln(x_2+c)\big)$$

with $c > \mathrm{e}$. Then clearly the first two limit conditions of (61) are satisfied. Moreover, for any $\lambda > 1$ (and thus for any $x^* > 3$) one has

$$\begin{aligned}\lim_{x_2\to\infty} \alpha(x_2)\mathrm{e}^{\lambda\beta(x_2)} &= \lim_{x_2\to\infty} \big(\ln(x_2+c)\big)^{-1}\mathrm{e}^{\lambda\ln\big(\ln(x_2+c)\big)}\\ &= \lim_{x_2\to\infty} \big(\ln(x_2+c)\big)^{\lambda-1} = \infty.\end{aligned}$$

Now with the above choice of α and β we can prove that Ω is "attractive". To this end, define

$$\begin{aligned}\Omega_1 &:= \big\{(x_1,x_2) \in \mathbb{R}^2_+ :\ f_1(x_1,x_2) < 0\big\}\\ &= \left\{(x_1,x_2) \in \mathbb{R}^2_+ :\ \tfrac{x_2^2}{x_2^2+1} < x_1\right\}\end{aligned}$$

and

$$\begin{aligned}\Omega_2 &:= \big\{(x_1,x_2) \in \mathbb{R}^2_+ :\ f_2(x_1,x_2) < 0\big\}\\ &= \left\{(x_1,x_2) \in \mathbb{R}^2_+ :\ x_1 < \tfrac{2x_2^2}{x_2^2+1}\right\}\end{aligned}$$

Case 1: Let $(x_1^0,x_2^0) \in \Omega_1 \setminus \Omega$. Then, $\Omega_1(x_1^0) := \Omega_1 \cap \{(x_1,x_2) \in \mathbb{R}^2_+ :\ x_1 \le x_1^0\}$ is positively invariant under the flow of (60). This follows easily from the behaviour of the vector field f on the boundary of $\Omega_f(x_1^0)$. On $\Omega_f(x_1^0)$, we can estimate f_2 as follows

$$\begin{aligned}|f_2(x_1,x_2)| &\le \left|\alpha(x_2)\Big(\mathrm{e}^{\beta(x_2)\big(x_1-\frac{2x_2^2}{x_2^2-1}\big)}-1\Big)\right|\\ &\le \big|\ln(x_2+c)\big|^{x_1} + \big|\ln(x_2+c)\big|^{-1}\\ &\le \big|\ln(x_2+c)\big|^{-1} + \big|\ln(x_2+c)\big|^{x_1^0}\\ &\le \big|\ln(x_2+c)\big|^{-1} + C(x_2^{1/x_1^0})^{x_1^0} \le C'x_2\end{aligned}$$

with appropriate constants $C > 0$ and $C' > 0$. Therefore, finite escape time phenomena can be excluded by a standard Grönwall type estimate. Hence, any solution starting in Ω_f has to reach Ω eventually.

Case 2: Let $(x_1^0,x_2^0) \in \Omega_2 \setminus \Omega$. Then, $\Omega_2(x_2^0) := \Omega_2 \cap \{(x_1,x_2) \in \mathbb{R}^2_+ :\ x_2 \le x_2^0\}$ is positively invariant under the flow of (60). Since $\Omega_2(x_2^0)$ is also bounded the corresponding solution does exist for all $t > 0$.

4.2.6. *Step 4.* Finally, from Figure 9 and the reasoning in Section 4.1 it is clear that this system does not have a max-separable Lyapunov function either.

5. **Conclusions.** This work has considered separable Lyapunov functions for monotone systems $\dot{x} = f(x)$ evolving on the positive orthant in Euclidean n-space. Here a Lyapunov function is called separable, if it can either be written as a sum or as a maximum of n functions, each of a single scalar argument. Due to a linearization argument and the Perron-Frobenius theory it is clear that both types of separable Lyapunov functions must exist in a small neighborhood of the origin, provided f is continuously differentiable at the origin and $Df(0)$ is Hurwitz.

Theorem 3.2 and its corollaries have shown that a max-separable Lyapunov function can be constructed for arbitrary compact sets in the domain of attraction, using essentially only the knowledge about one trajectory of the system.

For planar systems, Theorem 3.3 and its corollaries have clarified a necessary condition for max-separable Lyapunov functions to yield bounded level sets. Moreover, formulas have been given to construct sum-separable Lyapunov functions based on a little more information than one trajectory.

Both approaches have been demonstrated by examples, pointing out a discontinuity phenomenon and restrictions on the right hand side f that result from the existence of a certain type of Lyapunov function. It has also been argued in Section 3.2.3 that, despite their many advantages, sum-separable Lyapunov functions can be impractical for Lyapunov function based controller design due to the high-order nonlinearities that appear in our construction when the convergence rate of the system that can be obtained is too small (i.e., $c < 2$ in (29)). In contrast, a slow convergence rate does not necessarily imply high-order nonlinearities in the construction of max-separable Lyapunov functions, due to the "switching" between functions.

In the last section two elaborate examples were given, showing that there are systems that admit sum- but not max-separable Lyapunov functions and also globally asymptotically stable systems that admit neither.

Acknowledgement. The authors like to thank Michael Schönlein for his careful proofreading and many fruitful discussions.

REFERENCES

[1] D. Angeli and A. Astolfi, A tight small-gain theorem for not necessarily ISS systems, *Systems Control Lett.*, **56** (2007), 87–91.
[2] D. Angeli, E. D. Sontag and Y. Wang, A characterization of integral input-to-state stability, *IEEE Trans. Autom. Control*, **45** (2000), 1082–1097.
[3] A. Bacciotti, F. Ceragioli and L. Mazzi, Differential inclusions and monotonicity conditions for nonsmooth Lyapunov functions, *Set-Valued Analysis*, **8** (2000), 299–309.
[4] A. Berman and R. J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Academic Press, New York, 1979.
[5] N. P. Bhatia and G. P. Szegö, *Stability theory of dynamical systems*, Die Grundlehren der mathematischen Wissenschaften, Band 161, Springer-Verlag, New York, 1970.
[6] F. H. Clarke, Y. S. Ledyaev, R. J. Stern and P. R. Wolenski, *Nonsmooth analysis and control theory*, Springer, 1998.
[7] G. Como, E. Lovisari and K. Savla, Throughput optimality and overload behavior of dynamical flow networks under monotone distributed routing, *IEEE Trans. Contr. Network Systems*, To appear. doi:10.1109/TCNS.2014.2367361.
[8] S. N. Dashkovskiy, B. S. Rüffer and F. R. Wirth, Small gain theorems for large scale systems and construction of ISS Lyapunov functions, *SIAM J. Control Optim.*, **48** (2010), 4089–4118.
[9] P. De Leenheer, D. Angeli and E. D. Sontag, Monotone chemical reaction networks, *Journal of Mathematical Chemistry*, **41** (2007), 295–314.
[10] D. Hinrichsen and A. J. Pritchard, *Mathematical Systems Theory I — Modelling, State Space Analysis, Stability and Robustness*, Springer, Berlin, 2005.
[11] M. W. Hirsch and H. Smith, Monotone dynamical systems, in *Handbook of differential equations: ordinary differential equations. Vol. II*, Elsevier B. V., Amsterdam, 2005, 239–357.
[12] M. W. Hirsch, Systems of differential equations which are competitive or cooperative. I. Limit sets, *SIAM J. Math. Anal.*, **13** (1982), 167–179.

0bf24e7 2015-03-26 00:00:39 +0100 Accepted for publication in "Discrete and Continuous Dynamical Systems - Series B (DCDS-B)" p.27

[13] M. W. Hirsch, Systems of differential equations that are competitive or cooperative. II. Convergence almost everywhere, *SIAM J. Math. Anal.*, **16** (1985), 423–439.

[14] M. W. Hirsch, Systems of differential equations which are competitive or cooperative. III. Competing species, *Nonlinearity*, **1** (1988), 51–71.

[15] M. W. Hirsch, Systems of differential equations that are competitive or cooperative. V. Convergence in 3-dimensional systems, *J. Diff. Eqns.*, **80** (1989), 94–106.

[16] M. W. Hirsch, Systems of differential equations that are competitive or cooperative. IV. Structural stability in three-dimensional systems, *SIAM J. Math. Anal.*, **21** (1990), 1225–1234.

[17] M. W. Hirsch, Systems of differential equations that are competitive or cooperative. VI. A local C^r closing lemma for 3-dimensional systems, *Ergodic Theory Dynam. Systems*, **11** (1991), 443–454.

[18] M. W. Hirsch and H. L. Smith, Competitive and cooperative systems: mini-review, in *Positive systems (Rome, 2003)*, vol. 294 of Lecture Notes in Control and Inform. Sci., Springer, Berlin, 2003, 183–190.

[19] H. Ito, A Lyapunov approach to cascade interconnection of integral input-to-state stable systems, *IEEE Trans. Autom. Control*, **55** (2010), 702–708.

[20] H. Ito and Z.-P. Jiang, Necessary and sufficient small gain conditions for integral input-to-state stable systems: A Lyapunov perspective, *IEEE Trans. Autom. Control*, **54** (2009), 2389–2404.

[21] H. Ito, State-dependent scaling problems and stability of interconnected iISS and ISS systems, *IEEE Trans. Autom. Control*, **51** (2006), 1626–1643.

[22] H. Ito, S. Dashkovskiy and F. Wirth, Capability and limitation of max- and sum-type construction of Lyapunov functions for networks of iISS systems, *Automatica J. IFAC*, **48** (2012), 1197–1204.

[23] H. Ito, Z.-P. Jiang, S. Dashkovskiy and B. S. Rüffer, Robust stability of networks of iISS systems: Construction of sum-type Lyapunov functions, *IEEE Trans. Autom. Control*, **58** (2013), 1192–1207.

[24] H. Ito, B. S. Rüffer and A. Rantzer, Max- and sum-separable Lyapunov functions for monotone systems and their level sets, in *Proc. 53rd IEEE Conf. Decis. Control*, 2014, 2371–2377.

[25] Z.-P. Jiang, I. M. Y. Mareels and Y. Wang, A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems, *Automatica J. IFAC*, **32** (1996), 1211–1215.

[26] H. K. Khalil, *Nonlinear systems*, 3rd edition, Prentice-Hall, Upper Saddle River, NJ, USA, 2002.

[27] V. Lakshmikantham and S. Leela, *Differential and integral inequalities: Theory and applications. Vol. I: Ordinary differential equations*, Academic Press, New York, 1969.

[28] M. Margaliot and T. Tuller, Stability analysis of the ribosome flow model, *IEEE/ACM Trans. Comp. Biology & Bioinformatics*, **9** (2012), 1545–1552.

[29] A. Rantzer, Distributed control of positive systems, in *Proc. 50th IEEE Conf. Decis. Control and Europ. Contr. Conf.*, Orlando, FL, USA, 2011, 6608–6611.

[30] A. Rantzer, Optimizing positively dominated systems, in *Proc. 51st IEEE Conf. Decis. Control*, Maui, Hawaii, USA, 2012, 272–277.

[31] A. Rantzer, B. S. Rüffer and G. Dirr, Separable Lyapunov functions for monotone systems, in *Proc. 52nd IEEE Conf. Decis. Control*, 2013, 4590–4594.

[32] B. S. Rüffer, Monotone inequalities, dynamical systems, and paths in the positive orthant of Euclidean n-space, *Positivity*, **14** (2010), 257–283.

[33] B. S. Rüffer, Small-gain conditions and the comparison principle, *IEEE Trans. Autom. Control*, **55** (2010), 1732–1736.

[34] B. S. Rüffer, P. M. Dower and H. Ito, Computational comparison principles for large-scale system stability analysis, in *Proc. of the 10th SICE Annual Conference on Control Systems*, Kumamoto, Japan, 2010, (electronic).

[35] B. S. Rüffer, H. Ito and P. M. Dower, Computing asymptotic gains of large-scale interconnections, in *Proc. 49th IEEE Conf. Decis. Control*, 2010, 7413–7418.

[36] B. S. Rüffer, C. M. Kellett and S. R. Weller, Connection between cooperative positive systems and integral input-to-state stability of large-scale systems, *Automatica J. IFAC*, **46** (2010), 1019–1027.

[37] H. L. Smith, *Monotone dynamical systems*, vol. 41 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1995.

[38] E. D. Sontag, Input to state stability, in *The Control Systems Handbook: Control System Advanced Methods* (ed. W. S. Levine), 2nd edition, CRC Press, Boca Raton, 2010, 45.1–45.21 (1034–1054).

[39] E. D. Sontag, Smooth stabilization implies coprime factorization, *IEEE Trans. Autom. Control*, **34** (1989), 435–443.

[40] E. D. Sontag, Comments on integral variants of ISS, *Systems Control Lett.*, **34** (1998), 93–100.
[41] E. D. Sontag, Monotone and near-monotone biochemical networks, *Systems and Synthetic Biology*, **1** (2007), 59–87.
[42] E. D. Sontag and Y. Wang, On characterizations of the input-to-state stability property, *Systems Control Lett.*, **24** (1995), 351–359.
[43] J. A. Yorke, Extending Liapunov's second method to non-Lipschitz Liapunov functions, *Bull. Amer. Math. Soc.*, **74** (1968), 322–325.
[44] T. Yoshizawa, *Stability theory by Liapunov's second method*, Publications of the Mathematical Society of Japan, No. 9, The Mathematical Society of Japan, Tokyo, 1966.

Institut für Mathematik, Universität Würzburg, Campus Hubland Nord, Emil-Fischer-Str. 40, 97074 Würzburg, Germany
E-mail address: dirr@mathematik.uni-wuerzburg.de

Department of Systems Design and Informatics, Kyushu Institute of Technology, 680-4 Kawazu, Iizuka 820-8502, Japan
E-mail address: hiroshi@ces.kyutech.ac.jp

Automatic Control LTH, Lund University, Box 118, SE-221 00 Lund, Sweden
E-mail address: rantzer@control.lth.se

School of Mathematical & Physical Sciences, Faculty of Science & IT, The University of Newcastle (UoN), University Drive, Callaghan NSW 2308, Australia
E-mail address: bjoern@rueffer.info