A Two-Phase Approach to Stability of Networks Given in iISS Framework: Utilization of A Matrix-Like Criterion

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***Abstract*— This article is concerned with global asymptotic stability (GAS) of dynamical networks. The case when subsystems are integral input-to-state stable (iISS) has been recognized as notoriously difficult to deal with in the literature. In fact, the lack of energy dissipation for large input denies direct application of the small-gain argument for input-to-state stable (ISS) subsystems. Here for networks consisting of iISS subsystems it is demonstrated that a two-phase approach allows us to make use of the ISS small-gain argument by separating a trajectory into a transient and a subsequent ISS-like phase. In contrast to the previous iISS results, the two-phase approach immediately leads to a sufficiency criterion for GAS of general nonlinear networks, which is given in a matrix-like form (order condition).**

I. Introduction

Stability of equilibrium points is one of fundamental issues in analysis and design of complex dynamical systems. The notion of input-to-state stability (ISS) [22] is often useful in establishing global-type stability of nonlinear large-scale systems (networks) from their modules (subsystems). However, in practice one is often faced with subsystems exhibiting unbounded trajectories for inputs of finite magnitude even if their autonomous dynamics is stable. While such subsystems cannot be ISS, they may well be integral input-to-state stable (iISS) [2]. The class of iISS systems is strictly wider than that of ISS systems. Many practical examples suggest that a network can be stable even if it contains some iISS subsystems which are not ISS.

In recent years, small-gain type stability criteria have been developed successfully for networks in the framework of ISS [5], [6], [15], [14], [17]. However, the extension of these results to networks of iISS systems has been defied by the lack of instantaneous gain in the trajectory-based approach and the insufficiency of a popular maximization technique [12], [6], [17] for the construction of network Lyapunov functions [8], [20]. Only very recently, a solution to the network stability problem in the iISS framework has been given based on another type of network Lyapunov function in [11] without resorting to the maximization technique. Nevertheless, the stability criterion given in [11] does not reduce to the stability criterion given in a matrix-like form (order condition) [5], [6], [15] even when all subsystems are ISS. This discrepancy disappears when the number of subsystems is two [10]. For general numbers, the discrepancy has been closed only when subsystems are formulated in a particular way [11].

In this paper, without resorting to [11] and without constructing any network Lyapunov function, we aim at bridging the gap between the previous ISS and iISS network results by interpreting stability of networks composed of "iISS" subsystems in light of the "ISS" methodology. The natural approach taken in this paper separates the dynamics of the network into two phases. The first one is the transient phase. The second one is the subsequent ISS phase. The same approach has been employed in [9] for interconnected systems composed of two subsystems. A similar approach is also employed in [16] to solve an input-to-output stability problem without demanding ISS although the study does not formulate subsystems purely in the iISS framework given in [2]. Compared with [16], the method in [9] focuses on covering non-ISS subsystems by preserving the dissipation formulation and the small-gain type criterion developed previously for ISS subsystems. This paper continues to pursue this approach for general networks. The two phase methods have a drawback of being unable to deal with external signals [1], unless the magnitude of the external signals is sufficiently small [9], [16]. For the purposes of this paper, external inputs will not be considered. Due to space limitation, proofs are omitted[1].

In Section II we will recall some prerequisites, before we formally introduce the problem setup in Section III. The two phases of our decomposition procedure are described individually in Sections IV and V, and then combined in Section VI. Some examples are given in Section VII to illustrate the procedure, before we conclude in Section VIII.

II. Preliminaries

Let $\mathbb{R}_+$ be the interval $[0,\infty)$ in the set of real numbers $\mathbb{R}$. A continuous function $\omega:\mathbb{R}_+\to\mathbb{R}_+$ is said to be positive definite and written as $\omega\in\mathcal{P}$ if $\omega(0)=0$ and $\omega(s)>0$ for all $s\in(0,\infty)$. A function $\omega\in\mathcal{P}$ is said to be of class $\mathcal{K}$ and written as $\omega\in\mathcal{K}$ if it is strictly increasing. A function $\omega\in\mathcal{K}$ is of class $\mathcal{K}_\infty$ if $\lim_{s\to\infty}\omega(s)=\infty$. We write $\gamma\in\mathcal{K}\cup\{0\}$ to indicate that γ is either of class $\mathcal{K}$ or the zero function. The symbol $\mathbf{Id}$ denotes the identity map. For $\omega\in\mathcal{P}$, we write $\omega\in\mathcal{Q}$ if $\mathbf{Id}+\omega$ is strictly increasing. By definition, we have $\mathcal{P}\supset\mathcal{Q}\supset\mathcal{K}$. We have the following relationship:

Lemma 1: For any $\delta\in\mathcal{P}$, there exists $\tilde{\delta}\in\mathcal{Q}$ such that $s+\tilde{\delta}(s)\le s+\delta(s)$, $\forall s\in\mathbb{R}_+$.

Let $\overline{\mathbb{R}}_+:=[0,\infty]$. The inequalities $<$ and $\le$ on $\mathbb{R}_+$ are extended to $\overline{\mathbb{R}}_+$ with the convention $\infty\le\infty$. If γ is a

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[1]Proofs are available from the authors upon request.

class $\mathcal{K}_\infty$ function, its inverse γ^{-1} is of class $\mathcal{K}_\infty$. For $\gamma \in \mathcal{K} \setminus \mathcal{K}_\infty$, its inverse γ^{-1} is defined on the finite interval $[0, \lim_{\tau\to\infty}\gamma(\tau))$. For $\gamma \in \mathcal{K}$, an operator $\gamma^{\ominus}$: $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is defined as

$$\gamma^{\ominus}(s) = \sup\{v \in \mathbb{R}_+ : s \geq \gamma(v)\}.$$

We have $\gamma^{\ominus}(s) = \infty$ for $s \geq \lim_{\tau\to\infty}\gamma(\tau)$, and $\gamma^{\ominus}(s) = \gamma^{-1}(s)$ elsewhere. For a non-decreasing $\omega \in \mathcal{P}$, its extension ω: $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is

$$\omega(s) := \sup_{v \leq s} \omega(v).$$

Using these conventions for ω, $\gamma \in \mathcal{K}$, we have $\omega \circ \gamma^{\ominus}(s) = \lim_{\tau\to\infty}\omega(\tau)$ for $s \geq \lim_{\tau\to\infty}\gamma(\tau)$. The identity $\gamma^{\ominus} = \gamma^{-1} \in \mathcal{K}$ holds if and only if $\gamma \in \mathcal{K}_\infty$. It is important that, in the case of $\gamma \in \mathcal{K}\setminus\mathcal{K}_\infty$, we have only $\gamma\circ\gamma^{\ominus}(s) \leq s$ for $s \in \overline{\mathbb{R}}_+$ although $\gamma^{\ominus}\circ\gamma(s) = s$ for $s \in \overline{\mathbb{R}}_+$. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively.

For $a, b \in \overline{\mathbb{R}}^n$ the relation $a \geq b$ is defined by $a_i \geq b_i$ for all $i = 1, \ldots, n$. The negation of $a \geq b$ is denoted by $a \not\geq b$, i.e., there exists an $i \in \{1, \ldots, n\}$ such that $a_i < b_i$. The relation $a \gg b$ is defined by $a_i > b_i$ for all $i = 1, \ldots, n$. For notational convenience, we write $\infty = [\infty, \infty, ..., \infty]^T$, where the number of components is clear from the context.

III. Network of iISS systems

In this paper, we consider the network described by

$$\Sigma : \ \dot{x} = f(x), \tag{1}$$

where $f = [f_1^T, \ldots, f_n^T]^T : \mathbb{R}^N \to \mathbb{R}^N$ is locally Lipschitz and $f(0) = 0$. The state vector of Σ is $x = [x_1^T, \ldots, x_n^T]^T \in \mathbb{R}^N$, where $N := \sum_{i=1}^n N_i$. This paper addresses global asymptotic stability (GAS) of the equilibrium $x = 0$ of the network Σ. For convenience, we say $x = 0$ is asymptotically stable (AS) with respect to a set $\mathbf{D}$ if it is asymptotically stable and its region of attraction contains $\mathbf{D} \subset \mathbb{R}^N$. We suppose that there exist $\alpha_i \in \mathcal{K}$ and $\sigma_{ij} \in \mathcal{K}\cup\{0\}$ and positive definite, radially unbounded and continuously differentiable functions V_i: $\mathbb{R}^{N_i} \to \mathbb{R}_+$ for $i, j = 1, 2, ..., n$ such that the subsystems Σ_i, $i = 1, 2, ..., n$, in the form of

$$\dot{x}_i = f_i(x_1, \ldots, x_n), \quad x_i \in \mathbb{R}^{N_i} \tag{2}$$

satisfy

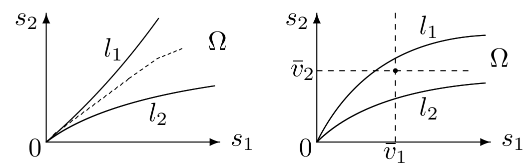
$$\frac{\partial V_i}{\partial x_i} f_i \leq -\alpha_i(V_i(x_i)) + \sum_{j=1}^n \sigma_{ij}(V_j(x_j)), \tag{3}$$

where $\sigma_{ii} = 0$ for $i = 1, 2, ..., n$.

The dissipation inequality (3) says that each subsystem Σ_i with the inputs x_j, $j \neq i$, is iISS, and that V_i is an iISS Lyapunov function for the disconnected subsystem Σ_i [2].

Let the operators $A, S, D : \mathbf{s} \in \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+^n$ be

$$A(\mathbf{s}) = [\alpha_1(s_1),\, \alpha_2(s_2),\, \ldots, \alpha_n(s_n)]^T$$
$$S(\mathbf{s}) = \left[\sum_{j=1}^n \sigma_{1,j}(s_j),\, \sum_{j=1}^n \sigma_{2,j}(s_j), \ldots, \sum_{j=1}^n \sigma_{n,j}(s_j)\right]^T$$
$$D(\mathbf{s}) = [s_1 + \delta_1(s_1), s_2 + \delta_2(s_2), \ldots, s_n + \delta_n(s_n)]^T.$$



(a) $\bar{v}_1$ and $\bar{v}_2$ are infinite. (b) $\bar{v}_1$ and $\bar{v}_2$ are finite.

Fig. 1. Two illustrative situations for (5) and (6).

The functions $\delta_i \in \mathcal{K}_\infty$ have yet to be determined.

Remark 1: Under the stronger assumption that $\alpha_i \in \mathcal{K}_\infty$, the subsystem Σ_i is guaranteed to be ISS, and the function V_i is an ISS Lyapunov function [23]. By definition [2], an ISS system is iISS. Note that the function V_i is qualified as an iISS Lyapunov function even when α_i is merely positive definite [2]. It is also important that the function V_i is qualified as an ISS Lyapunov function when $\lim_{s\to\infty}\alpha_i(s) \geq \lim_{s\to\infty}\sum_{j=1}^n \sigma_{ij}(s)$ [23].

IV. Phase 2: Stability in a Restricted Domain

Since the network in Section III is defined with subsystems which are not necessarily ISS, utilization of ISS tools requires us to give up $\mathbb{R}^N$ in analyzing the domain of attraction. Exploiting such tools, this section investigates stability of the network Σ in a subset $\mathbf{Z} \subseteq \mathbb{R}^N$. The domain-restricted stability secured in this section will be combined with another development in the next section to establish stability on the whole space $\mathbb{R}^N$. Define the following set:

$$\mathbf{Z}(\bar{\mathbf{v}}) = \{x \in \mathbb{R}^N : V_i(x_i) < \bar{v}_i,\ i = 1, 2, ..., n\}, \tag{4}$$

where $\bar{\mathbf{v}} = [\bar{v}_1, ..., \bar{v}_n]^T \in (0, \infty]^n$, and the components $\bar{v}_i \in (0, \infty]$, $i = 1, 2, ..., n$, have yet to be determined. Note that $\mathbf{Z}(\bar{\mathbf{v}}) = \mathbb{R}^N$ if $\bar{\mathbf{v}} = \infty$ is chosen. In order to deal with a network involving non-ISS subsystems, we are interested in making use of ISS tools only in a set $\mathbf{Z}(\bar{\mathbf{v}}) \neq \mathbb{R}_+^n$. It is not difficult to verify the following theorem by restricting the domain in the results of [21], [20], [4].

Theorem 1: Given $\bar{\mathbf{v}} \in (0, \infty]^n$, suppose that there exist $\delta_1, \delta_2, ..., \delta_n \in \mathcal{Q}$ satisfying

$$D \circ S(\mathbf{s}) \not\geq A(\mathbf{s}), \quad \forall \mathbf{s} \in \mathbb{R}_+^n \setminus \{0\} \tag{5}$$
$$D \circ S(\bar{\mathbf{v}}) \leq A(\bar{\mathbf{v}}) \tag{6}$$

Then the set $\mathbf{Z}(\bar{\mathbf{v}})$ is positively invariant with respect to (1), and the equilibrium $x = 0$ of Σ is AS with respect to $\mathbf{Z}(\bar{\mathbf{v}})$.

The operators $D \circ S$ and A are evaluated on the extended space $\overline{\mathbb{R}}_+^n$ in (6). An alternative expression of (6) is

$$\lim_{s\to\bar{v}_i} \alpha_i(s) \geq (\mathbf{Id} + \delta_i) \circ \sum_{j=1}^n \lim_{s\to\bar{v}_j} \sigma_{i,j}(s),\ i = 1, 2, ..., n.$$

The condition (6) can be interpreted in a geometrical picture used in [5], [6], [3], [20]. Define

$$\Omega = \{\mathbf{s} \in \mathbb{R}_+^n : -A(\mathbf{s}) + S(\mathbf{s}) \ll 0\}. \tag{7}$$

which is called a decay set in [6], [18], [20]. The fulfillment of (5) implies the existence of a non-empty Ω which is illustrated by Fig.1 in the $n = 2$ case. The boundary layer is given by the two curves l_1: $\alpha_1(s_1) = \sigma_{12}(s_2)$ and l_2:

$\alpha_2(s_2) = \sigma_{21}(s_1)$. The set $\Omega \neq \emptyset$ divides $\mathbb{R}^2_+ \setminus \{0\}$ into two disjoint sets. The existence of the functions δ_1 and δ_2 implies non-zero distance between the two curves l_1 and l_2 except at the origin. The vector $\bar{\mathbf{v}}$ belongs to Ω. Let $\bar{\mathbf{v}}$ be referred to as a decay point. Figure 1 (a) depicts a case where we can take $\bar{v}_1 = \bar{v}_2 = \infty$, while Figure 1 (b) illustrates (6) in the case where both $\bar{v}_1$ and $\bar{v}_2$ are finite. In fact, $\bar{v}_2$ needs to be finite in Figure 1 (b) since the decay set Ω is bounded in the s_2 component. This happens when Σ_1 is not ISS.

Remark 2: The condition (5) looks like a matrix condition. Indeed, it is verified in [11] that (5) is equivalent to

$$M(\mathbf{s}) \not\geq \mathbf{s}, \quad \forall \mathbf{s} \in \mathbb{R}^n_+ \setminus \{0\}, \tag{8}$$

where the operators $A^{\ominus}, M : \overline{\mathbb{R}}^n_+ \to \overline{\mathbb{R}}^n_+$ are defined by

$$M(\mathbf{s}) = A^{\ominus} \circ D \circ S(\mathbf{s}), \quad A^{\ominus}(\mathbf{s}) = \begin{bmatrix} \alpha_1^{\ominus}(s_1) \\ \alpha_2^{\ominus}(s_2) \\ \vdots \\ \alpha_n^{\ominus}(s_n) \end{bmatrix}.$$

In the case where A, S and D are linear, the operator M is represented by a matrix. Then the condition (8) is reduced to a spectral radius condition [3]. It is stressed that (6) is not equivalent to $M(\bar{\mathbf{v}}) \leq \bar{\mathbf{v}}$. In fact, $M(\bar{\mathbf{v}}) \leq \bar{\mathbf{v}}$ cannot guarantee $D \circ S(\bar{\mathbf{v}}) \leq A(\bar{\mathbf{v}})$ if $\bar{\mathbf{v}}$ contains ∞.

This paper aims at proving GAS of networks involving non-ISS subsystems by means of the matrix-like condition (5). The following holds true.

Proposition 1: Suppose that the origin $x = 0$ is GAS and that (5) holds. Then there exists a vector $\bar{\mathbf{v}} \in (0, \infty]^n$ satisfying (6). Moreover, for each $x(0) \in \mathbb{R}^N$, there exists a time $T_Z(x(0)) \geq 0$ such that $x(t)$ enters and remains in the set $\mathbf{Z}(\bar{\mathbf{v}})$ after a time $T_Z(x(0))$, i.e.,

$$x(t) \in \mathbf{Z}(\bar{\mathbf{v}}), \quad \forall t \geq T_Z(x(0)) \tag{9}$$

and $\lim_{t\to\infty} |x(t)| = 0$ for all $x(0) \in \mathbb{R}^N$.

In fact, the existence of $\bar{\mathbf{v}} \in (0, \infty]^n$ can be proved using [19], [18] with sufficient small $|\bar{\mathbf{v}}|$. The GAS ensures the finite-time convergence into $\mathbf{Z}(\bar{\mathbf{v}})$ for any trajectories of Σ. Theorem 1 guarantees that any trajectory entered $\mathbf{Z}(\bar{\mathbf{v}})$ remains there and moves toward $x = 0$. In this way, Proposition 1 states that the behavior of trajectories can be decomposed into two phases in explaining GAS of $x = 0$. Of course, we cannot assume the GAS in advance in practical situations. Therefore, to find a vector $\bar{\mathbf{v}} \in (0, \infty]^n$ achieving (6) and to prove the finite-time convergence into the corresponding set $\mathbf{Z}(\bar{\mathbf{v}})$ are the key two points in rendering the two-phase approach practically useful.

Remark 3: By virtue of Lemma 1, Theorem 1 can be stated with $\delta \in \mathcal{P}$. The use of $\delta \in \mathcal{Q}$ allows us to simplify presentations later on, since the monotonicity of M is ensured without referring to the existence of another $\delta \in \mathcal{Q}$ guaranteed by an original $\delta \in \mathcal{P}$.

Remark 4: In the case of $\bar{\mathbf{v}} = \infty$, the positive invariance of $\mathbf{Z}(\bar{\mathbf{v}})$ guaranteed by Theorem 1 implies the forward completeness of all solutions of (1). If $\alpha_i \in \mathcal{K}_\infty$ holds for all $i = 1, 2, ..., n$, the property (6) is satisfied for $\bar{\mathbf{v}} = \infty$. In fact, the vector $\bar{\mathbf{v}} = \infty$ satisfies (6) only if all subsystems are ISS [23]. It is also worth noting that the property (5) without the knowledge of (6) guarantees local asymptotic stability [19], [4] in the sense of a sufficiently small neighborhood of $x = 0$. Due to one of the fundamental ISS characterizations in [23], the property (6) implies that each subsystem Σ_i restricted to the domain $\mathbf{Z}(\bar{\mathbf{v}})$ is ISS with respect to input x_j ($j \neq i$) and state x_i. Theorem 1 relies on the ISS small-gain argument [13] in the domain $\mathbf{Z}(\bar{\mathbf{v}})$.

Remark 5: Theorem 1 does not require the functions α_i to be of class $\mathcal{K}_\infty$ in contrast to the previous ISS results [5], [6], [3], [18] which amount to $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$. Recall that all subsystems are ISS and the set Ω is radially unbounded [20], [8] when $\alpha_i \in \mathcal{K}_\infty$ holds for all $i = 1, 2, ..., n$. Note that the same condition, namely, that α_1, ..., $\alpha_n \in \mathcal{K}_\infty$ is implied by the assumptions in Theorem 3.17 of [20], i.e., the irreducibility and $\sigma_{ij} \in \mathcal{K}_\infty \setminus \{0\}$ for all i, j.

V. Phase 1: Transitional Period

The previous section has shown the idea of approaching the GAS for the iISS network Σ by separating each trajectory into two phases, and confirmed that the second phase is the convergence to $x = 0$ secured by the ISS small-gain regulation via (5). This section focuses on determining the switching point from the first phase to the second phase, and confirming that the first phase is a transitional period that can be linked successfully to the second phase. More precisely,

- **A.** an analytical formula for a decay point $\bar{\mathbf{v}} \in (0, \infty]^n$ satisfying (6) and
- **B.** finite-time convergence of all trajectories of (1) into the decay set $\mathbf{Z}(\bar{\mathbf{v}})$

are established in this section. Both, **A** and **B** cannot be treated as in the ISS case and hence require a different procedure. For attaining **A** and **B**, in this paper we do not want to introduce additional assumptions on the transitional period [16]. It is desirable to somehow make use of (5) again which is already used in the second phase. Adding the GAS criterion in [11] is against the purpose of this paper.

Consider $\tilde{\sigma}_{ij} \in \mathcal{K} \cup \{0\}$, $i, j = 1, 2, ..., n$, satisfying

$$\lim_{s\to\infty} \sum_{j=1}^{n} \tilde{\sigma}_{i,j}(s) > 0, \qquad i=1, 2, ..., n \tag{10}$$

$$\sigma_{i,j}(s) \leq \tilde{\sigma}_{i,j}(s), \quad \forall s \in \mathbb{R}_+, \quad i, j = 1, 2, ..., n \tag{11}$$

$$\tilde{\sigma}_{i,i}(s) = 0, \quad \forall s \in \mathbb{R}_+, \quad i = 1, 2, ..., n. \tag{12}$$

Define the operators $\tilde{M}, \tilde{D}, \tilde{S} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ as

$$\tilde{M}(\mathbf{s}) = A^{\ominus} \circ \tilde{D} \circ \tilde{S}(\mathbf{s})$$

$$\tilde{D}(\mathbf{s}) = [s_1 + \tilde{\delta}_1(s_1), s_2 + \tilde{\delta}_2(s_2), \ldots, s_n + \tilde{\delta}_n(s_n)]^T$$

$$\tilde{S}(\mathbf{s}) = \left[\sum_{j=1}^{n} \tilde{\sigma}_{1,j}(s_j), \sum_{j=1}^{n} \tilde{\sigma}_{2,j}(s_j), \ldots, \sum_{j=1}^{n} \tilde{\sigma}_{n,j}(s_j) \right]^T,$$

where $\tilde{\delta}_i \in \mathcal{Q}$, $i, j = 1, 2, ..., n$. Define $\mathbf{v}^{[h]} \in (0, \infty]^n$ as

$$\mathbf{v}^{[h]} = \begin{bmatrix} v_1^{[h]} \\ v_2^{[h]} \\ \vdots \\ v_n^{[h]} \end{bmatrix} = \tilde{M}^h(\infty) \tag{13}$$

for integers $h \geq 1$. We will make use of h. For $n = 2$, it is proved in [9] that the choice $\bar{\mathbf{v}} = \mathbf{v}^{[1]}$ achieves Items **A** and **B**. However, extending this fact to $n > 2$ with $\mathbf{v}^{[h]}$ is not easy in spite of the freedom h. In fact, there is a counter-example to the existence of h achieving **A** and **B** as shown in **Example 2** of Section VII. Therefore, this section aims at demonstrating when and how $\mathbf{v}^{[h]}$ in (13) achieves Items **A** and **B**. Note that we do not use M in defining $\mathbf{v}^{[h]}$. We have defined $\mathbf{v}^{[h]}$ using $\tilde{S}$ (i.e., $\tilde{\sigma}_{i,j}$) instead of S (i.e., $\sigma_{i,j}$). Property (10) guarantees that all components of $\mathbf{v}^{[h]}$ are not zero. Recall that the phase change point we want to obtain is $\bar{\mathbf{v}} \in (0, \infty]^n$. Let

$$\mathbf{B}(\mathbf{v}^{[h]}) = \{i \in \{1, 2, ..., n\} : v_i^{[h]} < \infty\}. \tag{14}$$

Then we can prove the following straightforwardly from the definition and the monotonicity of $\tilde{M}$ on $\overline{\mathbb{R}}_+^n$:

Lemma 2: It holds for all integers $h \geq 1$ that

$$\mathbf{v}^{[h]} \geq \mathbf{v}^{[h+1]} \tag{15}$$

$$\mathbf{B}(\mathbf{v}^{[h]}) \subseteq \mathbf{B}(\mathbf{v}^{[h+1]}). \tag{16}$$

$$\tilde{D} \circ \tilde{S}(\mathbf{v}^{[h]}) \leq A(\mathbf{v}^{[h]}) \ \Rightarrow\ \tilde{D} \circ \tilde{S}(\mathbf{v}^{[h+1]}) \leq A(\mathbf{v}^{[h+1]}) \tag{17}$$

$$\mathbf{v}^{[1]} = \infty \ \Leftrightarrow\ \mathbf{v}^{[h]} = \infty \tag{18}$$

$$\mathbf{B}(\mathbf{v}^{[1]}) = \emptyset \ \Leftrightarrow\ \mathbf{B}(\mathbf{v}^{[h]}) = \emptyset. \tag{19}$$

The next theorem shows that Item **B** is fulfilled.

Theorem 2: Let h be any integer $h \geq 1$. Assume that

$$\tilde{S} \circ \tilde{M}(\mathbf{s}) \ll \infty, \quad \forall \mathbf{s} \in \mathbb{R}_+^n \tag{20}$$

and (10), (11) and (12) are satisfied. If there exist $\tilde{\delta}_i \in \mathcal{Q}$, $i = 1, 2, ..., n$, such that

$$\tilde{M}(\mathbf{s}) \not\geq \mathbf{s}, \quad \forall \mathbf{s} \in \mathbb{R}_+^n \setminus \{0\} \tag{21}$$

and $\mathbf{B}(\mathbf{v}^{[h]}) \neq \emptyset$ hold, then $\mathbf{Z}(\mathbf{v}^{[h]})$ is positively invariant with respect to (1), and for each $x(0) \in \mathbb{R}^N$, there exists $T_Z(x(0)) \geq 0$ such that

$$x(t) \in \mathbf{Z}(\mathbf{v}^{[h]}), \quad \forall t \geq T_Z(x(0)) \tag{22}$$

is satisfied.

In order to address Item **A**, we define

$$\hat{M}(\mathbf{s}) = A^{\ominus} \circ \hat{D} \circ \tilde{S}(\mathbf{s}).$$

$$\hat{D}(\mathbf{s}) = [s_1 + \hat{\delta}_1(s_1), s_2 + \hat{\delta}_2(s_2), \ldots, s_n + \hat{\delta}_n(s_n)]^T$$

The following gives us a useful hint for attaining Item **A**:

Lemma 3: Assume that (20) holds, and that there exist $\hat{\delta}_i \in \mathcal{Q}$, $i = 1, 2, ..., n$, such that

$$\hat{M}(\mathbf{s}) \not\geq \mathbf{s}, \quad \forall \mathbf{s} \in \mathbb{R}_+^n \setminus \{0\}. \tag{23}$$

are satisfied. Let $\tilde{\delta}_i \in \mathcal{Q}$, $i = 1, 2, ..., n$, be such that

$$\tilde{\delta}_i(s) \leq \hat{\delta}_i(s), \quad \forall s \in \mathbb{R}_+, \quad i = 1, 2, ..., n \tag{24}$$

$$\tilde{\delta}_i(s) < \hat{\delta}_i(s), \quad \forall s \in \mathbb{R}_+ \setminus \{0\}, \quad i \in \mathbf{U} \tag{25}$$

where $\mathbf{U} = \{i \in \{1, 2, ..., n\} : \lim_{\tau\to\infty} \alpha_i(\tau) < \infty\}$. Then, for each $\mathbf{s} \in \mathbb{R}_+^n$, there exists an integer $h \geq 1$ such that

$$\tilde{M}^h(\mathbf{s}) \ll \infty. \tag{26}$$

It is also verified easily that (26) is achieved with $h = n$ if $\tilde{S}$ forms a cycle graph. Note that n is independent of $\mathbf{s}$. This fact is utilized for $n = 2$ in [9] to establish Items **A** and **B**. These observations suggest use of the following Lemma which certifies Item **A** for general graph topology.

Lemma 4: If there exists an integer $h \geq 1$ such that

$$\tilde{M}^h(\mathbf{s}) \ll \infty, \quad \forall \mathbf{s} \in \mathbb{R}_+^n, \tag{27}$$

then it holds that $\tilde{D} \circ \tilde{S}(\mathbf{v}^{[h]}) \leq A(\mathbf{v}^{[h]})$.

Note that the monotonicity of $\tilde{M}$ implies $\tilde{M}^{h+1}(\mathbf{s}) \ll \infty$ if $\tilde{M}^h(\mathbf{s}) \ll \infty$ holds. It is stressed that (26) is different from (27). In (26), the integer h may depends on $\mathbf{s}$, while (27) require h to be independent of $\mathbf{s}$. Based on Lemma 4, Item **A** can be achieved as follows:

Theorem 3: Let $\tilde{\delta}_i \in \mathcal{Q}$ and $\tilde{\sigma}_{ij} \in \mathcal{K} \cup \{0\}$, $i, j = 1, 2, ..., n$, be functions such that (10), (11), (12) and

$$\delta_i(s) \leq \tilde{\delta}_i(s), \quad \forall s \in \mathbb{R}_+, \quad i = 1, 2, ..., n \tag{28}$$

are satisfied. If there exists an integer $h \geq 1$ satisfying (27), then (6) holds with $\bar{\mathbf{v}} = \mathbf{v}^{[h]} \in (0, \infty]^n$.

The main message of Theorem 3 is that (6) can be derived for $\bar{\mathbf{v}}$ defined with (13) from (5) even when some subsystems are not ISS. In fact, if all subsystems are ISS with respect to all the inputs x_j $(j \neq i)$ in the sense of

$$\lim_{s\to\infty} \alpha_i(s) \geq (\mathbf{Id} + \tilde{\delta}_i) \circ \sum_{j=1}^{n} \lim_{s\to\infty} \tilde{\sigma}_{i,j}(s), \ i = 1, 2, ..., n, \tag{29}$$

then Theorem 3 holds for $\bar{\mathbf{v}} = \mathbf{v}^{[1]}$. Since the map $\tilde{M}$ is monotone on $\overline{\mathbb{R}}_+^n$, i.e., $x \leq y$ implies $\tilde{M}(x) \leq \tilde{M}(y)$, the property $\mathbf{v}^{[1]} \leq \infty$ implies $\tilde{M}(\mathbf{v}^{[1]}) \leq \tilde{M}(\infty)$, which is $\tilde{M}(\mathbf{v}^{[1]}) \leq \mathbf{v}^{[1]}$ by definition. If (29) holds, we have $A \circ \tilde{M} = \tilde{D} \circ \tilde{S}$ on $\overline{\mathbb{R}}_+^n$ and arrive at $\tilde{D} \circ \tilde{S}(\mathbf{v}^{[1]}) \leq A(\mathbf{v}^{[1]})$. Thus the choice $\bar{\mathbf{v}} = \mathbf{v}^{[1]}$ achieves (6). Here, the key is $A \circ \tilde{M}(\mathbf{s}) = \tilde{D} \circ \tilde{S}(\mathbf{s})$ which does not hold true for large $\mathbf{s}$ when some subsystems are not ISS, i.e., when (29) is not achieved. Therefore, in the absence of (29), the claim of Theorem 3 is not obvious.

VI. Merging Two Phases

We can establish GAS of Σ by combining the transient and the stability with domain restriction demonstrated in the previous sections. Theorems 1, 2 and 3 yield the following:

Theorem 4: If there exist functions $\delta_1, \delta_2, ..., \delta_n \in \mathcal{Q}$ and an integer $h \geq 1$ such that (5) and

$$M^h(\mathbf{s}) \ll \infty, \quad \forall \mathbf{s} \in \mathbb{R}_+^n \tag{30}$$

$$S \circ M(\mathbf{s}) \ll \infty, \quad \forall \mathbf{s} \in \mathbb{R}_+^n \tag{31}$$

are satisfied, then the equilibrium $x = 0$ of the network Σ is GAS.

Property (30) holds with $h = 1$ only if all subsystems are ISS. Choosing $h > 1$ allows us to deal with subsystems which are not ISS. The GAS in Theorem 4 is established by separating a trajectory into a transient and a subsequent convergent phase. In the case of $\mathbf{B}(\mathbf{v}^{[1]}) = \emptyset$, there is no transient period. Notice that, due to Lemma 2, $\mathbf{B}(\mathbf{v}^{[1]}) = \emptyset$ is equivalent to $\mathbf{B}(\mathbf{v}^{[h]}) = \emptyset$ for all integers $h \geq 1$.

Theorem 4 demonstrates that the matrix-like operator M gives a sufficient condition for GAS of the iISS network. It contrasts with the complete Lyapunov approach [11] which resulted in a stability criterion imposing a small-gain condition on all cycles in the network graph. When all subsystems are ISS and (5) holds with $\delta_1, \delta_2, ..., \delta_n \in \mathcal{Q}$, property (30) is guaranteed to hold with $h = 1$. This fact conform to the previous ISS result [6].

Robustness with respect to disturbances are not addressed by Theorem 4. The external signals invalidate Theorems 2 unless the effect of the disturbances is sufficiently small. This observation is reported in [9] for $n = 2$ of iISS systems, and in [16] for an input-to-output stability property. Such a constraint on disturbances could be considered as a natural requirement when we separate a trajectory into "the transient" and the rest of the trajectory.

Remark 6: Property (31) is implied by

$$\left\{ \lim_{s\to\infty} \alpha_j(s) = \infty \ \vee \ \lim_{s\to\infty} \sum_{i=1}^{n} \sigma_{i,j}(s) < \infty \right\}, j = 1, 2, ..., n. \quad (32)$$

In the case of $n = 2$, property (31) is guaranteed by (5). If (5) is satisfied for $n = 2$, property (30) also holds with $h = 2$. Hence, we can remove (30) and (31) completely if $n = 2$. Note that (32) is identical with the assumption made in the previous results for iISS networks [8], [11]. The previous results for ISS subsystems also require (32) [5], [6]. Networks of iISS subsystems satisfying (32) often arise in practical models by virtue of conservation through bounded nonlinearity. Some examples are the Monod equation and the Michaelis-Menten equation which are popular models of growth and reaction rates of microorganisms and enzymes. The rates are limited and described by bounded functions. Application of the mass balance to a biochemical reactor yields an interconnected system in which the decrease of one concentration or population results in the increase of another concentration, which amounts to (32) [7], [16].

VII. Examples

Example 1: Consider the network Σ specified by the following functions satisfying (31):

$$\alpha_1(s) = \frac{s}{1+s}, \ \alpha_2(s) = \frac{6s}{1+s}, \ \alpha_3(s) = \frac{6s}{1+s}$$
$$\sigma_{12}(s) = \frac{2s}{1+s}, \ \sigma_{21}(s) = \frac{s}{1+s}, \ \sigma_{23}(s) = \frac{s}{1+s}$$
$$\sigma_{31}(s) = \frac{s}{1+s}, \ \sigma_{11} = \sigma_{13} = \sigma_{22} = \sigma_{32} = \sigma_{33} = 0.$$

The number of subsystems is $n = 3$, and this network is formed by two coupled cycle graphs. Subsystem Σ_1 is not ISS, while the other subsystems are ISS. It is verified that (5) and (30) are achieved with $\delta_1(s) = \delta_2(s) = \delta_3(s) = 0.54s$ and $h = 2$. Theorem 4 guarantees that $x = 0$ of Σ is GAS. Now we compute a decay point $\bar{\mathbf{v}}$. Since S of Σ has no zero rows, we can set $\tilde{\sigma}_{i,j} = \sigma_{i,j}$, $i, j = 1, 2, 3$. Let $\tilde{\delta}_1(s) = \tilde{\delta}_2(s) = \tilde{\delta}_3(s) = 0.5s$. Equation (13) gives

$$\mathbf{v}^{[1]} = \begin{bmatrix} \infty & 1 & \frac{1}{3} \end{bmatrix}^T, \ \mathbf{v}^{[2]} = \begin{bmatrix} \infty & \frac{5}{11} & \frac{1}{3} \end{bmatrix}^T.$$

For the vector $\mathbf{v}^{[1]}$ we obtain

$$A(\mathbf{v}^{[1]}) - \tilde{D} \circ S(\mathbf{v}^{[1]}) = \begin{bmatrix} -\frac{1}{2} & \frac{9}{8} & 0 \end{bmatrix}^T \ngeq 0.$$

Thus, the property (6) is not satisfied with $\bar{\mathbf{v}} = \mathbf{v}^{[1]}$. However, it is satisfied with $\bar{\mathbf{v}} = \mathbf{v}^{[2]}$ since

$$A(\mathbf{v}^{[2]}) - \tilde{D} \circ S(\mathbf{v}^{[2]}) = \begin{bmatrix} \frac{5}{80} & 0 & 0 \end{bmatrix}^T \geq 0.$$

In fact, $\tilde{M}^2(\mathbf{s}) \ll \infty$ holds for all $\mathbf{s} \in \mathbb{R}_+^2$. Lemma 4 ensures that $\bar{\mathbf{v}} = \mathbf{v}^{[2]}$ achieves (6). In the set $\mathbf{Z}(\bar{\mathbf{v}})$ defined by (4) with $\bar{\mathbf{v}} = \mathbf{v}^{[2]}$, condition (5) serves as an ISS small-gain condition. In the outside region, condition (30) guarantees that any trajectory enters $\mathbf{Z}(\bar{\mathbf{v}})$ in finite time.

Example 2: Consider Σ of $n = 3$ specified by

$$\alpha_1(s) = 4s, \ \alpha_2(s) = 2s, \ \alpha_3(s) = \frac{2s}{1+s}$$
$$\sigma_{12}(s) = s, \ \sigma_{21}(s) = s. \ \sigma_{31}(s) = s$$
$$\sigma_{11} = \sigma_{13} = \sigma_{22} = \sigma_{23} = \sigma_{32} = \sigma_{33} = 0$$

which satisfies (31). This network consists of a cycle containing Σ_1 and Σ_2, and a cascade in which the state of Σ_1 is fed to Σ_3. The subsystem Σ_3 is not ISS. The condition (5) holds for $\delta_1(s) = \delta_2(s) = \delta_3(s) = ks$ if $k < 2\sqrt{2} - 1$. However, (30) does not hold for any $h \geq 1$. Since S of Σ does not contain zero rows, we set $\tilde{\sigma}_{i,j} = \sigma_{i,j}$, $i, j = 1, 2, 3$. Let $\tilde{\delta}_1(s) = \tilde{\delta}_2(s) = \tilde{\delta}_3(s) = s$. Then we obtain

$$\mathbf{v}^{[h]} = \begin{bmatrix} \infty & \infty & \infty \end{bmatrix}^T, \ h = 1, 2, ...$$

Since we have $A(\infty) = [\infty, \infty, 2]^T$ and $\tilde{D} \circ S(\infty) = [\infty, \infty, \infty]^T$, property (6) is not satisfied by $\mathbf{v}^{[h]}$ for any integer $h \geq 1$. In fact, for $\mathbf{s} = [a, b, c]^T$ we have

$$\tilde{M}^{2l+1}(\mathbf{s}) = \begin{bmatrix} \frac{b}{2^{l+1}} \\ \frac{a}{2^l} \\ \beta\left(\frac{a}{2^l}\right) \end{bmatrix}, \quad \tilde{M}^{2l+2}(\mathbf{s}) = \begin{bmatrix} \frac{a}{2^{l+1}} \\ \frac{b}{2^{l+1}} \\ \beta\left(\frac{b}{2^{l+1}}\right) \end{bmatrix}$$

for $l = 0, 1, 2, ...$, where

$$\beta(s) = \begin{cases} \dfrac{s}{2-s} & , \quad s \in [0, 2) \\ \infty & , \quad s \in [2, \infty) \end{cases}.$$

Hence, the integer h achieving (26) must depend on $\mathbf{s}$ as Lemma 3 suggests. This fact prevents us from using Lemma 4 to establish GAS of Σ. On the other hand, Σ satisfies

$$\alpha_1^{\ominus} \circ (\mathbf{Id} + \delta_1) \circ \sigma_{12} \circ \alpha_2^{\ominus} \circ (\mathbf{Id} + \delta_2) \circ \sigma_{21}(s) < s, \quad \forall s \in \mathbb{R}_+ \setminus \{0\}. \quad (33)$$

According to [11], property (33) guarantees that $x = 0$ of Σ is GAS. These facts reveal a case where a decay point $\bar{\mathbf{v}} \in \overline{\mathbb{R}}_+^n$ cannot be computed as simple as $\mathbf{v}^{[h]}$ in (13) although the network is actually GAS. In such a case, Proposition 1 suggests that one may find in a heuristic manner a vector $\bar{\mathbf{v}}$ achieving (6), i.e., here an example is $\bar{\mathbf{v}} = [1, 1, \infty]^T$. However, it is not clear how the finite-time convergence into the corresponding invariant set $\mathbf{Z}(\bar{\mathbf{v}})$ can be verified unless we introduce an additional assumption such as the one in [11]. Note that we do not have the equivalence between (5) and the small-gain condition in [11] for networks of general graph topology. In Example 2, the weak stability of a non-ISS subsystem is compensated by the strong stability of ISS subsystems, but none of subsystems has bounded ISS gain functions with respect to coupling inputs. In contrast, Example 1 allows us to invoke Lemma 4 since the network has ISS subsystems whose ISS gain function is bounded with respect to coupling inputs. In this way, condition (30) plays an important role in ensuring the practical usefulness of the two phase approach.

VIII. Concluding Remarks

This paper has given a two-phase interpretation to the mechanism of achieving GAS of a network in the presence of non-ISS subsystems. It provides an alternative to the recent result [11] on networks of iISS systems. The two phase approach cannot address stability with respect to external signals. Otherwise, the magnitude of external signals are required to be sufficiently small. Nevertheless, an advantage of the approach over the pure Lyapunov approach in [11] is that it leads to a stability criterion which takes the matrix-like form of the criterion developed previously for ISS networks [5], [6], [15]. The two phase interpretation can be given for iISS networks whenever the network is GAS. However, the two phase approach is not always practically useful, as was demonstrated in an example, where the proposed analytical formula cannot provide a phase change point. It is worth mentioning that the matrix-like criterion this paper focuses on was investigated in [20], and in the presence of non-ISS subsystems, the sufficiency of the matrix-like criterion for GAS remained unsolved there.

This paper has used (3) in defining dissipation inequalities of subsystems, which is sometimes called the summation aggregation [6]. Another typical way to formulate dissipation inequalities of subsystems is the maximization aggregation which replaces $\sum_{j=1}^{n}$ with $\max_{j\in\{1,2,\dots,n\}}$ in (3). It can be verified that all the results in this paper remain valid for the maximization aggregation. An additional favorable fact is that the maximization aggregation allows us to replace (26) with (27), i.e., h in Lemma 3 can be made independent of $\mathbf{s}$. Thus, (30) can be removed since it holds for sufficiently large h. It is remarkable that, for the maximization aggregation, without invoking the two phase approach, the authors have succeeded in proving the matrix-like sufficient condition to be sufficient for GAS of $x = 0$ of iISS networks in [11].

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