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Non-conservative dissipativity and small-gain theory for ISS networks

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***Abstract*— This paper addresses input-to-state stability (ISS) analysis for discrete-time systems using the notion of finite-step ISS Lyapunov functions. Here, finite-step Lyapunov functions are energy functions that decay after a fixed but finite number of steps, rather than at every time step. We establish non-conservative dissipativity and small-gain conditions for ISS of networks of discrete-time systems, by generalizing results in [1] and [2] to the case of ISS. The effectiveness of the results is illustrated through two examples.**

I. Introduction

There have been many contributions on stability analysis of large-scale systems over the last few decades, e.g., [3]–[7] to list just a few. However, it is still challenging to analyze the stability of interconnected systems with nonlinearities, and it is desirable to develop stability conditions which can be applied to a wide range of large-scale nonlinear systems. Dissipativity theory and small-gain theory are great tools to this end. The main idea with both is to split a large-scale system into smaller subsystems and analyze each subsystem individually. Then, if a suitable dissipativity or small-gain condition is satisfied, stability of the original system can be concluded. Unfortunately, in practice such a treatment is often conservative. This gave rise to the so-called non-conservative dissipativity and small-gain conditions proposed in [1], [2].

The non-conservative dissipativity and small-gain conditions primarily rely on the existence of a Lyapunov-like function called a finite-step Lyapunov function, which was originally introduced by Aeyels and Peuteman [8]. Such a function is not required to satisfy a dissipation inequality at each time step. Instead, it only needs to satisfy a dissipation-like inequality after some finite (but constant) time. Gielen and Lazar [1] studied global exponential stability (GES) for a feedback interconnection of discrete-time systems and developed dissipativity and small-gain conditions which are necessary and sufficient to assure GES for the interconnected system. The small-gain conditions in [1] were extended to the case of global asymptotic stability (GAS) by Geiselhart *et al.* [2].

In this paper, we first characterize input-to-state stability (ISS) analysis of discrete-time systems using so-called finite-step ISS Lyapunov functions, generalizing Theorem 2 in [8] to the case of ISS. Then we use this characterization to provide non-conservative dissipativity and small-gain conditions guaranteeing ISS for a network of discrete-time systems. These results do recover the dissipativity and small-gain conditions given in [1] and [2] when exogenous inputs are zero.

This paper is organized as follows: First relevant notation is recalled in Section II. Then the notion of finite-step ISS Lyapunov function for discrete-time systems is introduced in Section III. Non-conservative dissipativity and small-gain conditions for ISS of large-scale, discrete-time systems are provided in Section IV. Two illustrative examples are given in Section V. Section VI concludes the paper.

II. Notation

In this paper, $\mathbb{R}_{\geq 0}(\mathbb{R}_{>0})$ and $\mathbb{Z}_{\geq 0}(\mathbb{Z}_{>0})$ are the nonnegative (positive) real numbers and nonnegative (positive) integers, respectively. We denote the ith component of $v \in \mathbb{R}^n$ by v_i. For any $v, w \in \mathbb{R}^n$, we write $v \gg w$ ($v \geq w$) if and only if $v_i > w_i$ ($v_i \geq w_i$) for each $i \in \{1, \ldots, n\}$. If $v \geq w$ but $v \neq w$ we write $v > w$. A vector $v \in \mathbb{R}^n$ is *positive* if $v \gg 0$. The negation $v \not\geq w$ holds if and only if there exists some $i \in \{1, \ldots, n\}$ such that $v_i < w_i$.

For $v \in \mathbb{R}^n$, $\|v\|_p$ denotes the p-norm. The subscript p will be omitted in most places, but we assume that the same p-norm is used in all estimates.

Given a function $\varphi\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$, its sup-norm (possibly infinite) is denoted by $\|\varphi\|_{\ell^\infty} = \sup\{\|\varphi(k)\| : k \in \mathbb{Z}_{\geq 0}\} \leq \infty$. If no confusion arises, the sup-norm will again be denoted by $\|\cdot\|$, and the set of all functions $\mathbb{Z}_{\geq 0} \to \mathbb{R}^m$ with finite sup-norm is denoted by ℓ^∞.

A function $\alpha\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero and strictly increasing. It is of class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if $\alpha \in \mathcal{K}$ and also $\alpha(s) \to \infty$ as $s \to \infty$. A function $\beta\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$), if for each $s \geq 0$, $\beta(\cdot, s) \in \mathcal{K}$, and for each $r \geq 0$, $\beta(r, \cdot)$ is decreasing with $\beta(r, s) \to 0$ as $s \to \infty$. Let $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ and $\gamma_{iu} \in \mathcal{K} \cup \{0\}$ for each $i, j \in \{1, \ldots, n\}$ be given. We define $\overline{\Gamma}\colon \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ by

$$\overline{\Gamma}(s, z) := \begin{bmatrix} \max\left\{\gamma_{11}(s_1), \ldots, \gamma_{1N}(s_N), \gamma_{u1}(z)\right\} \\ \vdots \\ \max\left\{\gamma_{N1}(s_1), \ldots, \gamma_{NN}(s_N), \gamma_{uN}(z)\right\} \end{bmatrix}. \tag{1}$$

Also, let $\Gamma(s) := \overline{\Gamma}(s, 0)$ for all $s \in \mathbb{R}^n_{\geq 0}$. The identity function is denoted by id.

Composition of functions is denoted by the symbol $\circ$ and repeated composition of, e.g., a function γ by γ^i.

For $\alpha, \gamma \in \mathcal{K}$ we write $\alpha < \gamma$ if $\alpha(s) < \gamma(s)$ for all $s > 0$.

III. Finite-step ISS Lyapunov functions

This section provides a characterization of ISS for discrete-time systems via finite-step ISS Lyapunov functions.

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Consider the discrete-time system

$$\Sigma\colon x(k+1) = g(x(k), u(k)) \tag{2}$$

where $x(k) \in \mathbb{R}^n$, inputs or controls $u\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$ and $u \in \ell^\infty$. We assume that $g\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and $g(0,0) = 0$. For each $\xi \in \mathbb{R}^n$ and $u \in \ell^\infty$, $x(\cdot, \xi, u)$ denotes the trajectory of (2) with the initial value $x(0) = \xi$ and the input u. We will need the following definitions.

Definition 1 ([9]): The discrete-time system (2) is *input-to-state stable* (ISS) if there exist $\alpha \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that, for all $k \in \mathbb{Z}_{\geq 0}$, all $\xi \in \mathbb{R}^n$ and all $u \in \ell^\infty$, we have

$$\alpha(\|x(k,\xi,u)\|) \leq \max\left\{\beta(\|\xi\|, k), \sigma(\|u\|)\right\}.$$

Definition 2 ([9]): A continuous function $V\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an *ISS Lyapunov function* for (2) if there exist functions $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, $\gamma_u \in \mathcal{K}$, and $\alpha \in \mathcal{K}_\infty$ with $\alpha < \mathrm{id}$ such that for all $\xi \in \mathbb{R}^n$ and all $\mu \in \mathbb{R}^m$,

$$\underline{\alpha}(\|\xi\|) \leq V(\xi) \leq \overline{\alpha}(\|\xi\|), \text{ and} \tag{3}$$

$$V(g(\xi,\mu)) \leq \max\{\alpha(V(\xi)), \gamma_u(\|\mu\|)\}. \tag{4}$$

Definition 3: A continuous function $V\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a *finite-step ISS Lyapunov function* for (2) if there exist some $M \in \mathbb{Z}_{>0}$, functions $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, $\gamma_u \in \mathcal{K}$, and $\alpha \in \mathcal{K}_\infty$ with $\alpha < \mathrm{id}$ such that for all $\xi \in \mathbb{R}^n$ and all $u \in \ell^\infty$,

$$\underline{\alpha}(\|\xi\|) \leq V(\xi) \leq \overline{\alpha}(\|\xi\|), \text{ and} \tag{5}$$

$$V(x(M,\xi,u)) \leq \max\{\alpha(V(\xi)), \gamma_u(\|u\|)\}. \tag{6}$$

Now we are ready to state the first result of this paper.

Proposition 1: The following statements are equivalent:

(a) System (2) is ISS.
(b) System (2) admits a finite-step ISS Lyapunov function.

Proof: It is obvious that any ISS Lyapunov function is a finite-step ISS Lyapunov function. So the necessity follows from Theorem 1 in [9] showing that (2) is ISS if and only if it admits an ISS Lyapunov function. The sufficiency is more interesting. Pick some $\xi \in \mathbb{R}^n$ and an input $u \in \ell^\infty$. Let V be a finite-step ISS Lyapunov function for Σ. Let $\underline{\alpha}, \overline{\alpha}, \alpha$ and γ_u satisfying (5)–(6) be given. It follows from the fact that $\alpha < \mathrm{id}$, Remark 3.3 in [9] and Lemma 1 (see Appendix) that exists some $\tilde{\alpha} < \mathcal{K}_\infty$ with $\tilde{\alpha} < \mathrm{id}$ such that

$$V(x(M,\xi,u)) - V(\xi) \leq -\tilde{\alpha}(V(\xi)) + \gamma_u(\|u\|).$$

By a slight abuse of notation, we denote $\alpha := \tilde{\alpha}$. Pick some $\rho \in \mathcal{K}_\infty$ with $\rho < \mathrm{id}$. So we have

$$\begin{aligned} V(x(M,\xi,u)) - V(\xi) \leq &- (\mathrm{id} - \rho) \circ \alpha(V(\xi)) \\ &+ \gamma(\|u\|) - \rho \circ \alpha(V(\xi)). \end{aligned} \tag{7}$$

Let $b := \alpha^{-1} \circ \rho^{-1} \circ \gamma(\|u\|)$ and define $S := \left\{\xi \in \mathbb{R}^n\colon V(\xi) > b\right\}$. Obviously, $S^c := \mathbb{R}^n \backslash S$ is a compact set. It follows from (7) that for $\xi \in S$,

$$V(x(M,\xi,u)) - V(\xi) \leq -(\mathrm{id} - \rho) \circ \alpha(V(\xi))$$

Let $k_0 := k_0(\xi, u) := \min\left\{k \in \mathbb{Z}_{\geq 0}\colon x(k,\xi,u) \in S^c\right\} \leq \infty$. So by application of Lemma 3 (see Appendix), there exist some $\beta \in \mathcal{KL}$ and some real number $P > 1$ such that

$$V(x(k,\xi,u)) \leq \max_{i \in \{0,\ldots,M-1\}} P^i \beta(V(x(i,\xi,u)), k) \tag{8}$$

for all $k \in \{0, \ldots, k_0 - 1\}$. It follows from the continuity of $g(\cdot,\cdot)$ that there exists some $\kappa_1, \kappa_2 \in \mathcal{K}_\infty$ so that

$$\|g(\xi,\mu)\| \leq \kappa_1(\|\xi\|) + \kappa_2(\|\mu\|). \tag{9}$$

for all $\xi \in \mathbb{R}^n$ and all $\mu \in \mathbb{R}^m$. So for each $k < k_0$ we get

$$\begin{aligned} \|x(k+1,\xi,u)\| &\leq \kappa_1(\|x(k,\xi,u)\|) + \kappa_2(\|u(k)\|) \\ &\leq \kappa_1(\|x(k,\xi,u)\|) + \kappa_2(\|u\|) \\ &\leq \kappa_1(\|x(k,\xi,u)\|) + \kappa_2 \circ \gamma^{-1} \circ \rho \circ \alpha(b) \\ &\leq \kappa_1(\|x(k,\xi,u)\|) \\ &\quad + \kappa_2 \circ \gamma^{-1} \circ \rho \circ \alpha(V(x(k,\xi,u))) \\ &\leq \kappa_1(\|x(k,\xi,u)\|) \\ &\quad + \kappa_2 \circ \gamma^{-1} \circ \rho \circ \alpha \circ \overline{\alpha}(\|x(k,\xi,u)\|) \\ &=: \tilde{\kappa}(\|x(k,\xi,u)\|) \end{aligned}$$

It follows (5) that

$$\underline{\alpha}(\|x(k+1,\xi,u)\|) \leq \tilde{\kappa}(\|x(k,\xi,u)\|). \tag{10}$$

By applying (10) to (8) repeatedly, there exists some $\overline{\beta} \in \mathcal{KL}$ such that for all $k \in \{0, \ldots, k_0 - 1\}$

$$V(x(k,\xi,u)) \leq \overline{\beta}(V(\xi), k). \tag{11}$$

We claim that there exists some $\tilde{\sigma} \in \mathcal{K}$ such that for all $k \geq k_0$ we get

$$V(x(k,\xi,u)) \leq \tilde{\sigma}(b). \tag{12}$$

To see this, we note that

$$\begin{aligned} V(x(k_0+M,\xi,u)) &\leq -(\mathrm{id} - \rho) \circ \alpha(V(x(k_0,\xi,u))) \\ &\qquad + \gamma(\|u\|) \\ &\qquad + (\mathrm{id} - \rho \circ \alpha)(V(x(k_0,\xi,u))) \\ &\leq -(\mathrm{id} - \rho) \circ \alpha(V(x(k_0,\xi,u))) \\ &\qquad + \gamma(\|u\|) + (\mathrm{id} - \rho \circ \alpha)(b) \\ &= -(\mathrm{id} - \rho) \circ \alpha(V(x(k_0,\xi,u))) + b \\ &\leq b. \end{aligned} \tag{13}$$

This shows that (12) holds for all $k = k_0 + lM$, $l = 0, 1, \ldots$

Now assume that $M > 1$ and $V(x(k,\xi,u)) > b$ for all $k \geq k_0 + i + lM$, $i = 1, \ldots, M-1$, $l = 0, 1, \ldots$[1] So we have

$$\begin{aligned} V(x(k_0+M+1,\xi,u)) \leq &- (\mathrm{id} - \rho) \\ &\circ \alpha(V(x(k_0+1,\xi,u))). \end{aligned}$$

It follows by another application of Lemma 3 that

$$\begin{aligned} &V(x(k,\xi,u)) \leq \\ &\max_{i \in \{1,\ldots,M-1\}} P^{i-1} \beta(V(x(k_0+i,\xi,u), k - k_0 - 1) \end{aligned} \tag{14}$$

[1] Otherwise, $V(x(k+M,\xi,u)) \leq b$ holds for any $k \in \{k_0+1, \ldots, M-1\}$ for which $V(x(k,\xi,u)) \leq b$.

for all $k \geq k_0 + i + lM$, $i = 1, \dots, M-1$, and $l = 0, 1, \dots$ Without loss of generality, let $i = 1$ in (14). Also, let $\beta_0(s) := \beta(s, 0)$ for all $s \in \mathbb{R}_{\geq 0}$. Combination of (14) and (9), using the second inequality of (5) and monotonicity of $\beta, \overline{\alpha}, \kappa_1$ and κ_2 yield

$$\begin{aligned} V(x(k,\xi,u)) &\leq \beta_0(\overline{\alpha}(\kappa_1(\|x(k_0,\xi,u)\|) + \kappa_2(\|u\|))) \\ &\leq \beta_0(\overline{\alpha} \circ 2\kappa_1(\|x(k_0,\xi,u)\|) + \overline{\alpha} \circ 2\kappa_2(\|u\|)) \\ &\leq \beta_0 \circ 2\overline{\alpha} \circ 2\kappa_1 \circ \underline{\alpha}^{-1}(V(x(k_0,\xi,u))) \\ &\quad + \beta_0 \circ 2\overline{\alpha}(2\kappa_2(\|u\|)) \\ &\leq \beta_0 \circ 2\overline{\alpha} \circ 2\kappa_1 \circ \underline{\alpha}^{-1}(b) \\ &\quad + \beta_0 \circ 2\overline{\alpha} \circ 2\kappa_2 \circ \gamma^{-1} \circ \rho \circ \alpha(b) \\ &\leq \max\big\{\beta_0 \circ 2\overline{\alpha} \circ 2\kappa_1 \circ \underline{\alpha}^{-1}(b) \\ &\qquad + \beta_0 \circ 2\overline{\alpha} \circ 2\kappa_2 \circ \gamma^{-1} \circ \rho \circ \alpha(b), b\big\} \end{aligned} \tag{15}$$

for all $k \geq k_0 + i + lM, i = 1, \dots, M-1, l = 0, 1, \dots$. By repeating this procedure it follows that there exists some $\tilde{\sigma} \in \mathcal{K}$ such that

$$\begin{aligned} &V(x(k,\xi,u)) \leq \tilde{\sigma}(b) \\ &\quad \forall k \geq k_0 + i + lM, i = 1, \dots, M-1, l = 0, 1, \dots \end{aligned} \tag{16}$$

Combining (13) and (16) establishes (12). Combining (11) and (12) together with (5) we obtain the ISS estimate

$$\begin{aligned} &\underline{\alpha}(\|x(k,\xi,u)\|) \\ &\qquad \leq \max\big\{\overline{\beta}(V(\xi), k), \tilde{\sigma} \circ \rho^{-1} \circ \alpha^{-1} \circ \gamma(\|u\|)\big\} \\ &\qquad \leq \max\big\{\overline{\beta}(\overline{\alpha}(\|\xi\|), k), \tilde{\sigma} \circ \rho^{-1} \circ \alpha^{-1} \circ \gamma(\|u\|)\big\} \\ &\qquad \leq \max\big\{\tilde{\beta}(\|\xi\|, k), \sigma(\|u\|)\big\} \quad \forall k \in \mathbb{Z}_{\geq 0}. \end{aligned} \tag{17}$$

where $\tilde{\beta}(\cdot,\cdot) := \overline{\beta}(\overline{\alpha}(\cdot), \cdot)$ and $\sigma(\cdot) := \tilde{\sigma} \circ \rho^{-1} \circ \alpha^{-1} \circ \gamma(\cdot)$. This completes the proof. ∎

Remark 1: The implication from (b) to (a) extends [8, Thm. 2] to the case of ISS. Nešić *et al.* [10] provide a similar generalization to ISS systems for continuous-time systems.

Remark 2: We note that the continuity of $g(\cdot,\cdot)$ is not required to give the sufficiency, so one can replace continuity with a weaker assumption called $\mathcal{K}$-boundedness (see [2] and [11] for more details).

An alternative form of the dissipation formulation of ISS for discrete-time systems that does not require a continuous dynamics is provided in [12].

Further characterizations of input-to-state stability in case of a monotone dynamics can be found in [13].

IV. ISS NETWORKS OF SYSTEMS

This section applies Proposition 1 in order to establish ISS for a network of discrete-time systems. In particular, we show that similar non-conservative dissipativity and small-gain conditions to those in [1] and [2] yields input-to-state stability when the interconnected system is exposed to disturbances. To this end we assume that system (2) can be decomposed into N interconnected subsystems

$$\Sigma_i \colon x_i(k+1) = g_i(x(k), u(k)), \tag{18}$$

where $x_i(k) \in \mathbb{R}^{n_i}$ for each $i \in \{1, \dots, N\}$, inputs $u \colon \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$ with $u \in \ell^\infty$, $n := \sum_{i=1}^n n_i$, $x := [x_1^{\mathrm{T}}, \dots, x_N^{\mathrm{T}}]^{\mathrm{T}}$ and $g_i \colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n_i}$ with $g_i(0,0) = 0$. Let $g(\cdot,\cdot) := [g_1(\cdot,\cdot)^{\mathrm{T}}, \dots, g_N(\cdot,\cdot)^{\mathrm{T}}]^{\mathrm{T}}$ be continuous.

A. ISS from dissipativity conditions

We generalize the result in Theorem 2 of [1] in two ways. Firstly, we construct a storage function for Σ from the sum of linearly-weighted storage functions of individual subsystems. Secondly, we provide ISS for Σ rather than GES. We make the following assumption.

Assumption 1: Suppose that for each $i \in \{1, \dots, N\}$ there exist a set of storage and supply functions $V_i \colon \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ and $S_{ij} \colon \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \to \mathbb{R}$, $i \in \{1, \dots, N\}$, with $S_{ii} \equiv 0$ for each $i \in \{1, \dots, N\}$ such that the following hold:

1) There exist functions $\overline{\alpha}_i, \underline{\alpha}_i \in \mathcal{K}_\infty$ such that for all $\xi_i \in \mathbb{R}^{n_i}$,
$$\underline{\alpha}_i(\|\xi_i\|) \leq V_i(\xi_i) \leq \overline{\alpha}_i(\|\xi_i\|) \tag{19}$$
2) There exist some $M \in \mathbb{Z}_{>0}$, $\gamma \in [0,1)$ and $\gamma_{ui} \in \mathcal{K} \cup \{0\}$ such that for all $\xi \in \mathbb{R}^n$ and all $u \in \ell^\infty$ the dissipation-like inequality
$$\begin{aligned} &V_i(x_i(M,\xi,u)) \\ &\qquad \leq \max\left\{\gamma V_i(\xi_i) + \sum_{j=0}^{N} S_{ij}(\xi_i, \xi_j), \gamma_{ui}(\|u\|)\right\} \end{aligned} \tag{20}$$
holds, where $x_i(\cdot,\xi,u)$ denotes the trajectory of Σ_i corresponding to the initial value ξ and the input u.
3) There exists a positive vector $\lambda := [\lambda_1, \dots, \lambda_N]^{\mathrm{T}}$ with $\|\lambda\|_1 = 1$ such that for all $i, j \in \{1, \dots, N\}$,
$$\lambda_i S_{ij}(\xi_i, \xi_j) + \lambda_j S_{ji}(\xi_j, \xi_i) \leq 0. \tag{21}$$

Now we state sufficient conditions guaranteeing input-to-state stability for the composite system Σ.

Theorem 1: Under Assumption 1 the composite system Σ is ISS from u to x.

Proof: Define $W \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by

$$W(\xi) := \sum_{i=1}^{N} \lambda_i V_i(\xi_i).$$

Let $\xi \in \mathbb{R}^n$ and an input $u \in \ell^\infty$ be given. So from (20) we have

$$\begin{aligned} &W(x(M,\xi,u)) = \sum_{i=1}^{N} \lambda_i V_i(x_i(M,\xi,u)) \\ &\leq \sum_{i=1}^{N} \max\left\{\gamma\lambda_i V_i(\xi_i) + \lambda_i \sum_{j=1}^{N} S_{ij}(\xi_i,\xi_j), \lambda_i\gamma_{ui}(\|u\|)\right\} \\ &\qquad \leq \sum_{i=1}^{N} \lambda_i \left\{\gamma V_i(\xi_i) + \sum_{j=1}^{N} S_{ij}(\xi_i,\xi_j) + \gamma_{ui}(\|u\|)\right\} \end{aligned}$$

It follows from (21) that

$$\begin{aligned} W(x(M,\xi,u)) \quad &\leq \quad \sum_{i=1}^{N} \big\{\gamma\lambda_i V_i(\xi_i) + \gamma_{ui}(\|u\|)\big\} \\ &= \quad \gamma W(\xi) + \gamma_u(\|u\|). \end{aligned}$$

where $\gamma_u(\cdot) := \sum_{i=1}^N \lambda_i \gamma_{ui}(\cdot)$. From the proof of Proposition 1 it is straightforward to see that $W(\cdot)$ is a finite-step ISS Lyapunov function for Σ. ■

The following assumption is required to establish the converse of Theorem 1.

Assumption 2: Let V be an ISS Lyapunov function for the overall system Σ. Assume that $\underline{\alpha}$ and $\overline{\alpha}$ are functions of class-$\mathcal{K}_\infty$ satisfying (3). Also, let $\alpha \in \mathcal{K}_\infty$ with $\alpha < \mathrm{id}$ satisfy (4). Suppose that for some integer $M \geq 1$, some $c \in \mathbb{R}_{>0}$ and all $s \in \mathbb{R}_{>0}$ it holds that

$$\alpha^M(s) < \underline{\alpha} \circ \frac{1}{c}\,\mathrm{id} \circ \overline{\alpha}^{-1}(s). \tag{22}$$

Now we state the converse of Theorem 1.

Theorem 2: Let Σ be ISS. Suppose that Assumption 2 holds with $c = N^{\frac{1+p}{p}}$ where $p < \infty$ corresponds to the p-norm $\|\cdot\| = \|\cdot\|_p$. Then there exist functions $V_i \colon \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$, $S_{ij} \colon \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \to \mathbb{R}$, $\gamma_{ui} \in \mathcal{K} \cup \{0\}, i, j = 1, \ldots, N$, a positive vector $\lambda \in \mathbb{R}^N$ with $\|\lambda\|_1 = 1$ and $\gamma \in [0, 1)$ satisfying (19)–(21).

Proof: By Theorem 1 in [9], ISS for Σ implies the existence of an ISS Lyapunov function, say, $W \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying (3) and (4) with some $\alpha \in \mathcal{K}_\infty$ and $\gamma_u \in \mathcal{K}$. It follows from the fact that $\alpha < \mathrm{id}$ that

$$W(x(k,\xi,u)) \leq \max\{\alpha^k(W(\xi)), \gamma_u(\|u\|)\}. \tag{23}$$

Define $V_i \colon \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ and $S_{ij} \colon \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \to \mathbb{R}$ by

$$V_i(\xi_i) := \|\xi_i\| \qquad \forall i \in \{1,\ldots,N\} \tag{24}$$

$$S_{ij}(\xi_i,\xi_j) := \frac{\gamma}{N}\left(\|\xi_j\| - \|\xi_i\|\right) \qquad \forall i,j \in \{1,\ldots,N\}. \tag{25}$$

where $\gamma \in [0,1)$ which will be chosen later. Obviously, this choice of $V_i(\cdot)$ satisfies (19). Let $M \in \mathbb{Z}_{>0}$ satisfy (22) with $c = N^{\frac{1+p}{p}}$. So

$$\begin{aligned} V_i(x_i(M,\xi,u)) &= \|x_i(M,\xi,u)\| \leq \|x(M,\xi,u)\| \\ &\leq \underline{\alpha}^{-1}(W(x(M,\xi,u))) \end{aligned}$$

It follows from (23), the second inequality of (5) and monotonicity of $\underline{\alpha}^{-1}$ that

$$\begin{aligned} V_i(x_i(M,\xi,u)) &\leq \underline{\alpha}^{-1}(\max\{\alpha^M(W(\xi)), \gamma_u(\|u\|)\}) \\ &\leq \underline{\alpha}^{-1}(\max\{\alpha^M \circ \overline{\alpha}(\|\xi\|), \gamma_u(\|u\|)\}) \\ &= \max\big\{\underline{\alpha}^{-1} \circ \alpha^M \circ \overline{\alpha}(\|\xi\|), \\ &\qquad \underline{\alpha}^{-1} \circ \gamma_u(\|u\|)\big\} \end{aligned}$$

It follows with the fact that $\|\xi\| \leq N^{\frac{1}{p}} \max_j \|\xi_j\|_p$ that

$$\begin{aligned} V_i(x_i(M,\xi,u)) \leq \max\big\{ &\max_j\{\underline{\alpha}^{-1} \circ \alpha^M \circ \overline{\alpha}(N^{\frac{1}{p}}\|\xi_j\|)\}, \\ &\underline{\alpha}^{-1} \circ \gamma_u(\|u\|)\big\} \end{aligned}$$

By Assumption 2, we have

$$\underline{\alpha}^{-1} \circ \alpha^M \circ \overline{\alpha}(N^{\frac{1}{p}} s) < \frac{s}{N} \qquad \forall s > 0.$$

So there exists some $\gamma \in (0,1)$ such that

$$\begin{aligned} V_i(x_i(M,\xi,u)) &\leq \max\Big\{\max_j \frac{\gamma}{N}\|\xi_j\|, \underline{\alpha}^{-1} \circ \gamma_u(\|u\|)\Big\} \\ &\leq \max\Big\{\sum_{j=1}^N \frac{\gamma}{N}\|\xi_j\|, \underline{\alpha}^{-1} \circ \gamma_u(\|u\|)\Big\}. \end{aligned}$$

Denote $\tilde{\gamma}_u(\cdot) := \underline{\alpha}^{-1} \circ \gamma_u(\cdot)$. Also, from (24) and (25) we get $V_i(x_i(M,\xi,u)) \leq \max\Big\{\rho V_i(\xi_i) + \sum_{j=1}^N S_{ij}(\xi_i,\xi_j), \tilde{\gamma}_u(\|u\|)\Big\}$ which gives (20). Finally let $\lambda_i := \frac{1}{N}$ for all $i \in \{1,\ldots,N\}$; and so obviously $\|\lambda\|_1 = 1$. Moreover, this choice of λ together with S_{ij} for each $i, j \in \{1,\ldots,N\}$ satisfies (21). This completes the proof. ■

Remark 3: We note that it is shown in [2] that Assumption 2 is reasonable. In particular, [2] provides conditions under which Assumption 2 holds.

B. ISS from small-gain conditions

Next, we establish non-conservative small-gain conditions providing ISS for Σ, i.e., we extend Theorem 14 and Theorem 17 of [2] to the case of ISS. We first make the following assumption.

Assumption 3: Suppose that for each subsystem (18), $i \in \{1,\ldots,N\}$, there exists $V_i \colon \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ such that the following hold:

1) There exist functions $\overline{\alpha}_i, \underline{\alpha}_i \in \mathcal{K}_\infty$ such that for all $\xi_i \in \mathbb{R}^{n_i}$,

$$\underline{\alpha}_i(\|\xi_i\|) \leq V_i(\xi_i) \leq \overline{\alpha}_i(\|\xi_i\|). \tag{26}$$

2) There exist some $M \in \mathbb{Z}_{>0}$, $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ and $\gamma_{ui} \in \mathcal{K} \cup \{0\}$ such that for all $\xi \in \mathbb{R}^n$ and all $u \in \ell^\infty$ the dissipation-like inequality

$$\begin{aligned} &V_i(x_i(M,\xi,u)) \leq \\ &\max\Big\{\max_{j\in\{1,\ldots,N\}} \gamma_{ij}(V_j(\xi_j)), \gamma_{ui}(\|u\|)\Big\} \end{aligned} \tag{27}$$

holds, where $x_i(\cdot,\xi,u)$ denotes the trajectory of Σ_i corresponding to the initial value ξ and the input u.

The following definition is needed in the sequel.

Definition 4: The mapping Γ is said to satisfy the *small-gain condition* if

$$\Gamma(s) \not\geq s \tag{28}$$

holds for all $s \in \mathbb{R}^n_{\geq 0} \backslash \{0\}$.

Equivalently, as long as Γ is given by (1), cf. [6], (28) holds if all cycles in Γ are contractions, i.e.,

$$\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \cdot \circ \gamma_{i_{r-1} i_r} \circ \gamma_{i_r i_1} < \mathrm{id} \tag{29}$$

for all sequences $(i_1,\ldots,i_r) \in \{1,\ldots,N\}^r$ and all $r \geq 1$.

Theorem 3 (see below) provides sufficient conditions guaranteeing ISS for the overall system Σ. Later we add another condition to show that the converse holds as well.

Theorem 3: Let Assumption 3 and (28) hold. Then Σ is ISS from u to x.

Proof: Let $V_i \colon \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ for each $i \in \{1,\ldots,N\}$ satisfying Assumption 3 be given. Also, let $\xi \in \mathbb{R}^n$ and an input $u \in \ell^\infty$ be given. By Theorem 5.2 in [4], (28) implies that there exist functions $\sigma_i \in \mathcal{K}_\infty, i = 1, \ldots, N$ such that

$$\Gamma(\sigma(r)) \ll \sigma(r) \qquad \forall r > 0$$

where $\sigma(\cdot) = [\sigma_1(\cdot), \ldots, \sigma_N(\cdot)]^{\mathrm{T}}$. So we have

$$\sigma_i^{-1}(\max\{\gamma_{i1} \circ \sigma_1(r), \ldots, \gamma_{iN} \circ \sigma_N(r)\}) < r \tag{30}$$

for all $r > 0$, all $i \in \{1, \ldots, N\}$. Let $\alpha(\cdot) := \max_{i,j} \sigma_i^{-1} \circ \gamma_{ij} \circ \sigma_j(\cdot)$. It follows from (30) that $\alpha < \mathrm{id}$. Define $W\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by

$$W(\xi) := \max_i \psi_i(\xi_i).$$

where $\psi_i(\xi_i) := \sigma_i^{-1}(V_i(\xi_i))$ for all $\xi_i \in \mathbb{R}^{n_i}$. Let $W(\cdot)$ be a finite-step ISS Lyapunov function candidate for Σ. For space reason, by a slight abuse of notation let $x(M) = x(M, \xi, u)$. So we have

$$\begin{aligned} W(x(M)) &= \max_i \sigma_i^{-1}(V_i(x_i(M))) \\ &\leq \max_{i,j} \sigma_i^{-1}(\max\{\gamma_{ij}(V_j(\xi_j)), \gamma_{ui}(\|u\|)\}) \\ &= \max_{i,j} \sigma_i^{-1}(\max\{\gamma_{ij} \circ \sigma_j(\psi_j(\xi_j)), \gamma_{ui}(\|u\|)\}) \\ &\leq \max_{i,j,l} \sigma_i^{-1}(\max\{\gamma_{ij} \circ \sigma_j(\psi_l(\xi_l)), \gamma_{ui}(\|u\|)\}) \\ &= \max_{i,j} \sigma_i^{-1}(\max\{\gamma_{ij} \circ \sigma_j(W(\xi)), \gamma_{ui}(\|u\|)\}) \\ &= \max\{\alpha(W(\xi)), \gamma_u(\|u\|)\}. \end{aligned} \tag{31}$$

where $\gamma_u(\cdot) := \max_i \sigma_i^{-1} \circ \gamma_{ui}(\cdot)$. The inequality (31) implies that $W(\cdot)$ is indeed a finite-step ISS Lyapunov function for Σ. ■

Now we state the converse of Theorem 3.

Theorem 4: Let Σ be ISS. Also, let Assumption 2 hold with $c = N^{\frac{1}{p}}$ where $p < \infty$ corresponds to the p-norm. Then there exist functions $V_i\colon \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$, $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$, and $\gamma_{ui} \in \mathcal{K} \cup \{0\}, i, j = 1, \ldots, N$ satisfying (26)-(27). Also, the functions $\gamma_{ij}, i, j = 1, \ldots, N$ satisfy (28).

Proof: By Theorem 1 in [9], ISS for Σ implies the existence of an ISS Lyapunov function, say, $W\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying (3) and (4) with some $\alpha \in \mathcal{K}_\infty$ and $\gamma_u \in \mathcal{K}$. It follows from the fact that $\alpha < \mathrm{id}$ that

$$W(x(k, \xi, u)) \leq \max\{\alpha^k(W(\xi)), \gamma_u(\|u\|)\}. \tag{32}$$

Define $V_i\colon \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ by

$$V_i(\xi_i) := \|\xi_i\| \qquad \forall i \in \{1, \ldots, N\}. \tag{33}$$

Obviously, this choice of $V_i(\cdot)$ satisfies (26). Let $M \in \mathbb{Z}_{>0}$ satisfy (22) with $c = N^{\frac{1}{p}}$. So

$$\begin{aligned} V_i(x_i(M, \xi, u)) &= \|x_i(M, \xi, u)\| \leq \|x(M, \xi, u)\| \\ &\leq \underline{\alpha}^{-1}(W(x(M, \xi, u))) \end{aligned}$$

It follows from (32) that

$$\begin{aligned} V_i(x_i(M, \xi, u)) &\leq \underline{\alpha}^{-1}(\max\{\alpha^M(W(\xi)), \gamma_u(\|u\|)\}) \\ &\leq \underline{\alpha}^{-1}(\max\{\alpha^M \circ \overline{\alpha}(\|\xi\|), \gamma_u(\|u\|)\}) \\ &= \max\{\underline{\alpha}^{-1} \circ \alpha^M \circ \overline{\alpha}(\|\xi\|), \\ &\qquad \underline{\alpha}^{-1} \circ \gamma_u(\|u\|)\} \end{aligned}$$

From the fact that $\|\xi\| \leq N^{\frac{1}{p}} \max_j \|\xi_j\|_p$ and (33), we get

$$V_i(x_i(M, \xi, u)) \leq \max\{\max_j \gamma_x(V_j(\xi_j)), \underline{\alpha}^{-1} \circ \gamma_u(\|u\|)\}$$

where $\gamma_x(\cdot) := \underline{\alpha}^{-1} \circ \alpha^M \circ \overline{\alpha} \circ N^{\frac{1}{p}} \mathrm{id}(\cdot)$. It follows from Assumption 2 that $\gamma_x(s) < s$ for all $s > 0$. Let $\gamma_{ij}(.) := \gamma_x(.)$ for all $i, j \in \{1, \ldots, N\}$ and $\tilde{\gamma}_u(\cdot) := \underline{\alpha}^{-1} \circ \gamma_u(\cdot)$. This implies that $V_i(x_i(M, \xi, u)) \leq \max\{\max_j \gamma_{ij}(V_j(\xi_j)), \tilde{\gamma}_u(\|u\|)\}$; and so (27) holds. Finally the fact that $\gamma_{ij} < \mathrm{id}$ for all $i, j \in \{1, \ldots, N\}$ gives (28). This completes the proof. ■

Remark 4: Before the submission of the final version of this paper the authors became aware that similar non-conservative small-gain conditions have almost concurrently been reported in [14].

V. ILLUSTRATIVE EXAMPLES

Two examples are provided to illustrate the effectiveness of the results in the previous sections.

A. Example 1

Consider the discrete-time system

$$\Sigma\colon \begin{cases} x_1(k+1) = x_1(k) + x_2(k) + u_1(k) \\ x_2(k+1) = -x_2(k) - x_1(k) + u_2(k) \end{cases} \tag{34}$$

where $x_i(k), u_i(k) \in \mathbb{R}$ for each $i \in \{1, 2\}$. We consider the inputs $u := [u_1, u_2]^{\mathrm{T}}$ as disturbances. We aim to establish ISS of the system (34) using the non-conservative dissipativity conditions. Split the system (34) into two subsystems. Let $V_i(\xi_i) = \xi_i^2$ for $i = 1, 2$ be a storage function candidate for the ith subsystem. Such a function satisfies (19). However, it is straightforward to see that the storage function candidate $V_i, i = 1, 2$ fails to satisfy (20) with $\gamma \in [0, 1)$ when $M = 1$ and so the classical dissipation theory is not applicable. Interestingly, for $M = 2$ we have

$$\begin{aligned} V_1(x_1(2, \xi, u)) &= (u_1(0) + u_2(0) + u_1(1))^2 \leq 9\|u\|^2 \\ V_2(x_2(2, \xi, u)) &= (-u_1(0) - u_2(0) + u_2(1))^2 \leq 9\|u\|^2 \end{aligned}$$

So (20) holds with $\gamma = 0, S_{ij} = 0$ and $\gamma_{ui}(s) = 9s^2$ for all $i, j \in \{1, 2\}$. By the application of Theorem 1, the system (34) is ISS.

B. Example 2

Consider the discrete-time system

$$\Sigma\colon \begin{cases} x_1(k+1) = x_1(k) + 0.8x_2(k) \\ x_2(k+1) = -0.4x_1(k) + 0.2x_2(k) + u(k) \end{cases} \tag{35}$$

where $x_i(k) \in \mathbb{R}$ for each $i \in \{1, 2\}$ and $u(k) \in \mathbb{R}$. We want to show ISS for the system (35) by the non-conservative small-gain conditions. Split the system (35) into two subsystems. It should be pointed out that the first subsystem is not ISS when it is decoupled from the second subsystem. So we cannot get an ISS Lyapunov function for the first subsystem, although the whole system is ISS. This implies that the classical small-gain conditions (see Theorem 3 in [9] for example) fail to prove ISS for the system (35).

We note that the solution to Σ_2 from the initial value $\xi = [\xi_1, \xi_2]^{\mathrm{T}}$ and the input u at $M = 3$ is

$$\begin{aligned} x_1(3,\xi,u) =& 0.296\xi_1 + 0.736\xi_2 + 0.96u(0) + 0.8u(1) \\ x_2(3,\xi,u) =& -0.44\xi_2 - 0.368\xi_1 - 0.28u(0) + 0.2u(1) \\ &+ u(2) \end{aligned}$$

Let $V_i(\xi_i) = \|\xi_i\|$ for $i = 1, 2$. This choice of V_i satisfies (26) for some $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$. So we get

$$\begin{aligned} V_1(x_1(2,\xi,u)) &\le 0.296\,\|\xi_1\| + 0.736\,\|\xi_2\| + \|u\| \\ &\le \max\{0.592V_1(\xi_1), 1.472V_2(\xi_2), 2\,\|u\|\} \\ V_2(x_2(2,\xi,u)) &\le 0.368\,\|\xi_1\| + 0.44\,\|\xi_2\| + 0.3\,\|u\| \\ &\le \max\{0.736V_1(\xi_1), 0.88V_2(\xi_2), 0.6\,\|u\|\} \end{aligned}$$

This gives (27) with the gain functions

$$\begin{aligned} \gamma_{11}(s) &= 0.592s, \quad \gamma_{12}(s) = 1.472s, \quad \gamma_{u1} = 2s, \\ \gamma_{21}(s) &= 0.736s, \quad \gamma_{22}(s) = 0.88s, \quad \gamma_{u2} = 0.6s. \end{aligned}$$

We emphasize that $\max\{\gamma_{11}, \gamma_{22}\} < \mathrm{id}$. Moreover, we note that $\gamma_{12} \circ \gamma_{21} < \mathrm{id}$. So by the cycle condition (29) the small-gain condition (28) holds as well. Hence, we conclude ISS for the system (35) from Theorem 3.

VI. Conclusions

A characterization of input-to-state stability for discrete-time systems using the notion of finite-step ISS Lyapunov functions was given. Here, Lyapunov-like functions satisfy a dissipation inequality after some finite number of steps rather than after each time step. This characterization was employed to study ISS for large-scale interconnected system. Particularly, non-conservative dissipativity and small-gain conditions have been provided. The theoretical results have been illustrated by two examples.

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Appendix

We begin with establishing an equivalence between two seemingly different Lyapunov characterizations of ISS. This equivalence is used in the proof of Proposition 1.

Lemma 1: Let $g\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be continuous with $g(0,0) = 0$. Then the following statements are equivalent:

1) There exists a smooth function $V\colon \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ positive definite and $\alpha < \mathrm{id}$, $\gamma \in \mathcal{K}$, such that for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \text{ and} \tag{36}$$

$$V(g(x,u)) \le \max\big\{\alpha(V(x)), \gamma(\|u\|)\big\}. \tag{37}$$

2) There exists a smooth function $V\colon \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\overline{\alpha}\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ positive definite, $\overline{\gamma} \in \mathcal{K}$, such that for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \text{ and} \tag{38}$$

$$\begin{aligned} V(g(x,u)) - V(x) &\le -\overline{\alpha}(V(x)) \\ \text{whenever } V(x) &\ge \overline{\gamma}(\|u\|). \end{aligned} \tag{39}$$

Moreover, in both formulations the functions α and γ, respectively, $\overline{\alpha}$ and $\overline{\gamma}$ may be assumed to be of class $\mathcal{K}_\infty$.

Proof: We start with the last statement. Obviously, α and γ in (37) can be enlarged to unbounded (and strictly increasing) functions without changing the nature of the characterization. By [9, Remark 3.3] the corresponding functions in (39) may also be assumed to be of class $\mathcal{K}_\infty$.

Now assume that characterization 1 holds. Without loss of generality, we assume that $\alpha \in \mathcal{K}_\infty$. Fix $x \in \mathbb{R}^n$, $x \neq 0$, and $u \in \mathbb{R}^m$. If $V(x) \ge \alpha^{-1} \circ \gamma(\|u\|)$ then by (37) we have

$$\begin{aligned} & V(g(x,u)) \le \alpha(V(x)) \\ \iff & V(g(x,u)) - V(x) \le -(\mathrm{id} - \alpha)(V(x)) \\ & \qquad =: -\overline{\alpha}(V(x)) < 0, \end{aligned}$$

where $\overline{\alpha}$ is clearly positive definite, which together with (36) shows that characterization 2 holds as well.

Finally, assume that characterization 2 is given. For $s \in \mathbb{R}_{\ge 0}$ define

$$\tilde{\gamma}(s) := \max\Big\{\sup\big\{V(g(x,u))\colon V(x) \le \overline{\gamma}(s), \|u\| \le s\big\}, \overline{\gamma}(s)\Big\},$$

a positive definite, non-decreasing and unbounded function. By standard techniques we can bound $\tilde{\gamma}$ from above by a $\gamma \in \mathcal{K}_\infty$.

Now assume that $V(x) \geq \gamma(\|u\|)$ with $x \neq 0$. Then also $V(x) \geq \overline{\gamma}(\|u\|)$ as γ over-bounds $\overline{\gamma}$. Hence from (39) we get

$$\begin{aligned} &V(g(x,u)) - V(x) \leq -\overline{\alpha}(V(x)) < 0 \\ \iff &V(g(x,u)) \leq V(x) - \overline{\alpha}(V(x)) = (\mathrm{id} - \overline{\alpha})(V(x)) \\ &\qquad =: \alpha(V(x)) < V(x) \end{aligned}$$

The function α is clearly nonnegative with $\alpha(0) = 0$ can therefore be bounded from above by a positive definite function, which we denote again by α. In addition, α satisfies $\alpha < \mathrm{id}$.

Next assume that $V(x) \leq \gamma(\|u\|)$. Then by definition of γ we have that also $V(g(x,u)) \leq \gamma(\|u\|)$.

Combining the last two cases, we find that $V(g(x,u)) \leq \max\{\alpha(V(x)), \gamma(\|u\|)\}$. Moreover, (38) immediately gives (36). This completes the proof. ∎

Definition 5 ([10]): Let $\beta \in \mathcal{KL}$ be given. The function $\beta(\cdot,\cdot)$ is said to be uniformly incrementally bounded (UIB) if there exists some $P > 1$ such that $\beta(s,\tau) \leq P\beta(s,\tau+1)$ for all $s \in \mathbb{R}_{\geq 0}$ and all $\tau \in \mathbb{Z}_{\geq 0}$.

We note that Lemma 1 in [10] shows that any function of class-$\mathcal{KL}$ can be majorized by a UIB function. Using this lemma we have the following result:

Lemma 2 ([10]): Let a UIB function $\beta \in \mathcal{KL}$ and an integer $l \geq 0$ be given. There exists some $P > 1$ such that for any $s \in \mathbb{R}_{\geq 0}$ and any $\tau \in \mathbb{Z}_{\geq 0}$

$$\beta(s,\tau) \leq P^l \beta(s, \tau + l).$$

Lemma 3: Let a function $V\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be given. Assume that there exists some $M \in \mathbb{Z}_{>0}$ and $\alpha \in \mathcal{K}_\infty$ with $\alpha(s) < s$ for all $s \in \mathbb{R}_{>0}$ such that

$$V(k+M) - V(k) \leq -\alpha(V(k)) \qquad \forall k \geq k_0, k_0 \in \mathbb{Z}_{\geq 0}.$$

Then there exists some $\beta \in \mathcal{KL}$ and some real number $P > 1$ such that we have

$$V(k) \leq \max_{i \in \{0,\dots,M-1\}} P^i \beta(V(k_0 + i), k - k_0) \qquad \forall k \geq k_0.$$

Proof: Without loss of generality we assume that $\alpha < \frac{\mathrm{Id}}{M}$. Define a function $y_i\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$y_i(t) := V(k) + (t-k)(V(k+M) - V(k)) \tag{40}$$

for all $t \in [k, k+M]$, all $k \in \{k_0 + i, k_0 + M + i, \dots\}$, and $i \in \{0, \dots, M-1\}$. Note that the function $y_i(\cdot)$ is absolutely continuous. We also emphasize that $y_i(k) = V(k)$ for all $k = k_0 + i + lM$, $l \in \mathbb{Z}_{\geq 0}$. So for almost all $t \geq k_0 + i$ we have

$$\dot{y}_i(t) = V(k+M) - V(k) \leq -\alpha \circ V(k) \leq -\alpha \circ y_i(t). \tag{41}$$

It follows from the standard comparison lemma for differential equations that there exists some $\beta \in \mathcal{KL}$ such that the following holds

$$y_i(t) \leq \beta(y_0, t - t_0). \tag{42}$$

where $t_0 = k_0 + i, y_0 = V(k_0 + i)$ and $i \in \{0, \dots, M-1\}$. Using the facts that $y_i(k) = V(k), y_0 = V(k_0 + i)$ and $t_0 = k_0 + i$, for each $k \in \{k_0 + i, k_0 + i + M, \dots\}$ and $i \in \{0, \dots, M-1\}$ we get

$$V(k) \leq \beta(V(k_0 + i), k - k_0 - i). \tag{43}$$

Without loss of generality, we assume that $\beta(\cdot,\cdot)$ is UIB, for otherwise, by Lemma 1 in [10], it can be majorized with some UIB function $\bar{\beta} \in \mathcal{KL}$. By application of Lemma 2, for any $i \in \{0, \dots, M-1\}$ there exists some $P > 1$ such that for any $V(k_0 + i) \in \mathbb{R}_{\geq 0}$ and $k \in \{k_0 + i, k_0 + M + i, \dots\}$ the following holds

$$V(k) \leq P^i \beta(V(k_0 + i), k - k_0). \tag{44}$$

So we have

$$V(k) \leq \max_{i \in \{0,\dots,M-1\}} P^i \beta(V(k_0 + i), k - k_0) \tag{45}$$

for all $k \geq k_0$. This completes the proof. ∎