Connection between cooperative positive systems and integral input-to-state stability of large-scale systems☆

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Abstract

We consider a class of continuous-time cooperative systems evolving on the positive orthant $\mathbb{R}^n_+$. We show that if the origin is globally attractive, then it is also globally stable and, furthermore, there exists an unbounded invariant manifold where trajectories strictly decay. This leads to a small-gain type condition which is sufficient for global asymptotic stability (GAS) of the origin.

We establish the following connection to large-scale interconnections of (integral) input-to-state stable (ISS) subsystems: If the cooperative system is (integral) ISS, and arises as a comparison system associated with a large-scale interconnection of (i)ISS systems, then the composite nominal system is also (i)ISS. We provide a criterion in terms of a Lyapunov function for (integral) input-to-state stability of the comparison system. Furthermore, we show that if a small-gain condition holds then the classes of systems participating in the large-scale interconnection are restricted in the sense that certain iISS systems cannot occur. Moreover, this small-gain condition is essentially the same as the one obtained previously by Dashkovskiy, Rüffer, and Wirth (2007, 2009b) for large-scale interconnections of ISS systems.

Key words: nonlinear systems, dissipation inequalities, comparison system, monotone systems, integral input-to-state stability (iISS), Lyapunov function, small-gain condition, nonlinear gain
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1. Introduction

Consider $n \geq 1$ control systems of the form

$$\Sigma_i : \; \dot{x}_i = f_i(x_1, \ldots, x_n, u_i), \quad i = 1, \ldots, n, \tag{1}$$

where $x_i \in \mathbb{R}^{N_i}$, $u_i \in \mathbb{R}^{M_i}$, $N = \sum N_i$, $M = \sum M_i$, $f_i : \mathbb{R}^{N+M_i} \to \mathbb{R}^{N_i}$ is locally Lipschitz with $f_i(0) = 0$, satisfying dissipative integral input-to-state stability estimates

$$\langle \nabla V_i(x_i), f_i(x, u_i)\rangle \leq -\alpha_i(V_i(x_i)) + \sum_{j \neq i} \gamma_{ij}(V_j(x_j)) + \gamma_{iu}(\|u_i\|) , \tag{2}$$

for all $x_j \in \mathbb{R}^{N_j}$, $j = 1, \ldots, n$, and $u_i \in \mathbb{R}^{M_i}$, where each $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ is assumed to be continuously differentiable, such that

$$\underline{\alpha}_i(\|x_i\|) \leq V_i(x_i) \leq \overline{\alpha}_i(\|x_i\|) , \quad \text{for all } x_i \in \mathbb{R}^{N_i}, \tag{3}$$

for some $\mathcal{K}_\infty$ functions $\underline{\alpha}_i, \overline{\alpha}_i$, and the functions $\alpha_i, \gamma_{ij}, \gamma_{iu}$ are assumed to be locally Lipschitz continuous. The functions γ_{ij} and γ_{iu} are called *gains* and assumed to be of class $\mathcal{G} = \mathcal{K} \cup \{0\}$, i.e., they are each either class $\mathcal{K}$ functions or zero. Throughout we assume that $\gamma_{ii} = 0$. The functions α_i are assumed to be positive definite.

If in addition a function α_i is in class $\mathcal{K}_\infty$, then the corresponding system Σ_i is in fact input-to-state stable (ISS). It is known that an arbitrary composition of ISS systems is ISS, provided a small-gain condition is satisfied (Dashkovskiy et al., 2007, 2009b). Here we will treat the more general iISS case (Sontag, 1998).

There exist several conditions in the literature (Arcak, Angeli, and Sontag, 2002; Ito, 2006; Chaillet and Angeli, 2008) for the stability of the composite system

$$\Sigma : \; \dot{x} = f(x, u) , \tag{4}$$

with $x = (x_1^T, \ldots, x_n^T)^T$, $u = (u_1^T, \ldots, u_n^T)^T$, and $f(x, u) = (f_1(x, u_1)^T, \ldots, f_n(x, u_n)^T)^T$, arising by treating $(\Sigma_1, \ldots, \Sigma_n)$ as one single system under different forms of structural assumptions on the interconnection graph structure. Central to all existing results that are based on the input-to-state stability concept are growth and scaling conditions, which can be quite intricate.

In general neither cascades nor feedback loops of iISS systems yield stable systems. Ito (2006) gave stability conditions for feedback loops of two iISS systems in terms of a small-gain condition and scaling conditions, together with a recipe for the construction of a Lyapunov function for the composite system. Chaillet and Angeli (2008) have treated the case of cascaded iISS systems in detail. Here the only necessary condition is a scaling condition. Similar scaling conditions have also been used before by Arcak et al. (2002) to design robust

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output-feedback control laws. In a recent contribution, which was only brought to our attention after the submission of this paper, Dashkovskiy, Ito, and Wirth (2009a) have extended the work Dashkovskiy et al. (2009b) to a dissipative Lyapunov formulation, utilizing small-gain arguments as well as what could be called generalized left-eigenvectors for the construction of Lyapunov functions.

In this paper, we use a comparison principle approach involving a vector Lyapunov function, which naturally arises as the vector of the Lyapunov functions of the subsystems. The resulting comparison system is a positive, cooperative system. We study its dynamics and geometric implications of global asymptotic stability (GAS), which on the converse side lead us to a small-gain type condition. We find other sufficient conditions in terms of Lyapunov functions that indicate when the comparison system is not only GAS but also (i)ISS. The small-gain condition implies the existence of an unbounded path in a decay set. We show that the existence of such a path is incompatible with certain classes of supply rates, which could be thought of as "pure" iISS.

For the sake of a simpler exposition in this paper, the right-hand sides of estimates (2) (the so-called *supply rates*) are not given in terms of norms of the states but in terms of Lyapunov functions of the states. This will ease notation dramatically. Moreover, we only treat the time-invariant case, but the time-varying case is a straightforward extension.

The idea of using a comparison system to deduce stability properties of a *nominal system* or "the object of inquiry" is not new. An excellent recent overview of the available results in comparison theory can be found in Michel, Hou, and Liu (2008). Our approach is to aggregate several types of existing results: Comparison techniques as detailed by Lakshmikantham and Leela (1969a,b) and results on monotone dynamical systems by Smith (1995) for the comparison system induced by the interconnection topology, as well as a monotone selection theorem (Rüffer, 2009; Dashkovskiy et al., 2009b).

We use a nonlinear matrix-vector type formulation, by defining operators A, Γ and $G : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ per

$$A(v)_i = \alpha_i(v_i)\,, \quad \Gamma(v)_i = \sum_{j\neq i} \gamma_{ij}(v_j)\,, \quad G(w)_i = \gamma_{iu}(w_i)\,, \quad (5)$$

for $i = 1, \ldots, n$. The general idea is that stability properties of the comparison system

$$\dot{v} = -A(v) + \Gamma(v) + G(w), \qquad v, w \in \mathbb{R}^n_+\,, \quad (6)$$

induced by the right-hand sides of the dissipation inequalities (2) translate into the same stability properties of the composite system (4).

This paper is organized as follows. In Section 2 we recall some necessary definitions, in particular different formulations of input-to-state stability and related properties. Section 3 contains the main results of the paper, starting with comparison principles for (i)ISS and GAS of large-scale systems in Section 3.1. This is followed by topological, i.e., geometrical, implications of GAS of the origin with respect to the autonomous part of (6), namely an existence result for an invariant decay set in Section 3.2. This naturally leads to a small-gain condition, which in a strengthened form is used in a sufficiency criterion in Section 3.3. Here we also see that one of the implications of the small-gain condition, which is the existence of an unbounded path in the decay set, has implications for the possible supply pairs. Section 4 concludes the paper.

2. Preliminaries

In this section we establish some necessary notation. The *positive orthant* $\mathbb{R}^n_+$ in $\mathbb{R}^n$ is the set $\{x \in \mathbb{R}^n : x_i \geq 0\, \forall i\}$. By the boundary of $\mathbb{R}^n_+$, also denoted $\partial\mathbb{R}^n_+$, we mean the set $\{s \in \mathbb{R}^n_+ : \exists i : s_i = 0\}$. In $\mathbb{R}^n$ the open ball of radius $r > 0$ centered at x is denoted by $B(x, r)$. The p-norm on $\mathbb{R}^n$ is denoted by $\|\cdot\|_p$, where p is usually omitted in the case $p = 2$. The max-norm is denoted as $\|\cdot\|_\infty$. The inner product on $\mathbb{R}^n$ is denoted by $\langle x, y\rangle = x^T y$ for $x, y \in \mathbb{R}^n$. The sphere of radius $r \geq 0$ with respect to the 1-norm, intersected with the positive orthant $\mathbb{R}^n_+$, is an $(n-1)$-*simplex* and denoted by $S_r := \{x \in \mathbb{R}^n_+ : \|x\|_1 = r\}$.

The *order* on $\mathbb{R}^n$ is given by $x \leq y$ if and only if $x_i \leq y_i$ for all i; $x < y$ if and only if $x \leq y$ and $x \neq y$; and $x \ll y$ if and only if $x_i < y_i$ for all i. Notably, the condition $x \not\geq y$ is not the same as $x < y$ but rather indicates the existence of at least one component i, such that $x_i < y_i$. In other words, $x \not\geq y$ means: Either $x < y$ or x and y are not comparable. In particular, we will use the notation $M \not\geq 0$ for operators $M : \mathbb{R}^n_+ \to \mathbb{R}^n$ to denote that $M(v) \not\geq 0$ for all $v \in \mathbb{R}^n_+$, $v \neq 0$. A set $\Omega \subset \mathbb{R}^n_+$ is called *radially unbounded* if for any $v \in \mathbb{R}^n_+$, there exists a $w \in \Omega$ satisfying $v \leq w$.

The *comparison function* classes $\mathcal{K}$ and $\mathcal{K}_\infty$ are, respectively, the sets of continuous functions $\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+\,, \gamma(0) = 0, \gamma \text{ is strictly increasing}\}$ and $\{\gamma \in \mathcal{K} : \gamma \text{ is unbounded}\}$. For short we write class $\mathcal{G} = \mathcal{K} \cup \{0\}$ to include the zero function. The class of continuous positive definite functions $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is denoted by $\mathcal{PD}$. A function $\beta : \mathbb{R}^2_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if for fixed $t \geq 0$ the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for fixed $s \geq 0$ the function $\beta(s, \cdot)$ is non-increasing with $\lim_{t\to\infty} \beta(s, t) = 0$.

A *finite directed graph* G is a pair (V, E) of a set of vertices V and directed edges $E \subset V \times V$. Usually we will identify $V = \{1, \ldots, n\}$ for some $n \geq 1$. A *path* of length k is a sequence of edges $((i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k))$ with $(i_j, i_{j+1}) \in E$ for all $j = 1, \ldots, k$. A *cycle* is a path with $i_1 = i_k$, i.e., the initial and terminal vertices coincide. A graph is *strongly connected* if for any pair of vertices i, j there is a path from vertex i to vertex j and a path from vertex j to vertex i. The adjacency matrix $A_G = (a_{ij}) \in \{0, 1\}^{n\times n}$ of G is defined as

$$a_{ij} = \begin{cases} 1 & \text{if } e_{ji} \in E \\ 0 & \text{otherwise.} \end{cases}$$

The matrix A_G is *irreducible* iff G is strongly connected and *reducible* otherwise (Berman and Plemmons, 1979).

Similarly, any $n \times n$ matrix $\Gamma = (\gamma_{ij})$ induces a directed graph G_Γ, where we set $V = \{1, \ldots, n\}$ and define $E \subset V \times V$ per $(j, i) \in E \iff \gamma_{ij} \neq 0$. Note that $(j, i) \in E$ does not automatically imply $(i, j) \in E$, i.e., edges defined this way are directed. We will call Γ irreducible, if G_Γ is strongly connected and reducible otherwise. In particular, we will think of the nonlinear operator Γ defined in (5) as a matrix with entries that are functions, $\Gamma = (\gamma_{ij})$, for that matter.

Note that this directed graph notion is compatible with the signal flow diagram of the network of interconnected systems (1) and corresponds to the graph of the *gain matrix* of the network, i.e., the matrix $\Gamma = (\gamma_{ij})$ consisting of the gains γ_{ij} in (2).

2.1. Input-to-state type stability concepts

We consider a system

$$\dot{x} = f(x, u) \tag{7}$$

satisfying the usual Carathéodory assumptions on uniqueness and local existence of solutions, with $x \in \mathbb{R}^N$ and $u \in \mathbb{R}^M$. Let $V : \mathbb{R}^N \to \mathbb{R}_+$ be a continuously differentiable function for which there exist two $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}$, such that the estimate

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|) \tag{8}$$

holds for all $x \in \mathbb{R}^N$. Recall that $\langle \nabla V(x), f(x, u)\rangle$ denotes the derivative of V along trajectories of (7). Such a function V is called a Lyapunov function candidate.

If there exist $\alpha, \gamma \in \mathcal{K}_\infty$, such that the dissipation inequality

$$\langle \nabla V(x), f(x, u)\rangle \leq -\alpha(\|x\|) + \gamma(\|u\|) \tag{9}$$

holds, then system (7) is called *input-to-state stable* (ISS) (see, e.g., Sontag and Teel, 1995) and V is termed an *ISS Lyapunov function* (in the *dissipative formulation*). Other equivalent formulations of ISS exist, and they include trajectory estimates, asymptotic gain properties combined with local stability (Sontag and Wang, 1996, 1995) or input-to-state dynamical stability (ISDS, Grüne, 2002a,b). A related, but not equivalent, concept is differential input-to-state stability (Angeli, Sontag, and Wang, 2003). An excellent overview of the “big picture” on ISS can be found in Sontag (2001). In addition there exist at least two more equivalent formulations involving Lyapunov functions: ISDS, and the following so-called implication form. The *implication form* requires a Lyapunov function candidate and a gain $\gamma \in \mathcal{K}$ such that the implication

$$\|x\| > \gamma(\|u\|) \implies \langle \nabla V(x), f(x, u)\rangle < 0$$

holds for all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}^M$. Observe that in all formulations qualitatively, due to (8), we could have replaced $\|x\|$ with $V(x)$. Doing so will simplify our notation significantly.

For brevity we say a system of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^N \tag{10}$$

is GAS, if the origin is *globally asymptotically stable*, i.e., for all $x \in \mathbb{R}^N$, the solution $\Phi(t; x)$ exists for all $t \geq 0$, $\lim_{t\to\infty} \|\Phi(t; x)\| = 0$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x\| < \delta$ implies $\|\Phi(t; x)\| < \varepsilon$ for all $t \geq 0$. Recall (e.g., from Lin, Sontag, and Wang, 1996, Proposition 2.5) that GAS is equivalent to the existence of a class-$\mathcal{KL}$ function β, such that

$$\|\Phi(t; x)\| \leq \beta(\|x\|, t), \text{ for all } t \geq 0.$$

Similarly, we say system (7) is 0-GAS if it is GAS for $u \equiv 0$. It is well known (e.g., Teel and Praly, 2000, Corollary 2) that if f in (10) is locally Lipschitz then GAS is equivalent to the existence of a smooth Lyapunov function $V : \mathbb{R}^N \to \mathbb{R}_+$ satisfying (8) and

$$\langle \nabla V(x), f(x, u)\rangle \leq -V(x)\,.$$

It is the dissipative formulation of ISS (9) which extends easily to a more general case. Given functions $\alpha, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a Lyapunov function candidate the dissipation estimate $\langle \nabla V(x), f(x, u)\rangle \leq -\alpha(V(x)) + \gamma(\|u\|)$ holds for all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}^M$, system (7) is termed *input-to-state stable* (ISS) if $\alpha \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$. Observe the slightly weaker requirement on γ which is equivalent to the definition given above and is preferred by some authors. The system is termed *integral input-to-state stable* (iISS) if $\alpha \in \mathcal{PD}$ and $\gamma \in \mathcal{K}$. In particular, this includes ISS as a special case.

The pair (α, γ) is called a *supply pair*, the function γ is called the *supply function* or *gain*. The function α is termed the *decay rate*.

Remark 2.1. *The difference between ISS and iISS may seem very subtle at first. The ISS property might be interpreted as an L^∞ to L^∞ stability property, but it is also equivalent to a form of L^2 to L^2 stability. In contrast, iISS is more of an L^2 to L^∞ stability property (Sontag, 2001): The equivalent trajectory formulation is that there exists a $\mathcal{KL}$ function β and functions $\underline{\alpha}, \gamma \in \mathcal{K}$ such that for all $x^0 \in \mathbb{R}^N$, all $t \geq 0$, and all locally integrable inputs $u : \mathbb{R}_+ \to \mathbb{R}^M$, $\underline{\alpha}(\|x(t; x^0)\|) \leq \beta(\|x^0\|, t) + \int_0^t \gamma(\|u(s)\|)ds$.*

In the literature, iISS as above in the dissipative Lyapunov formulation is often defined with $\gamma \in \mathcal{K}$, but sometimes also using supply pairs where γ is of class $\mathcal{K}_\infty$ (Arcak et al., 2002; Ito, 2006; Chaillet and Angeli, 2008; Ito, 2008; Angeli et al., 2000a; Sontag, 2001; Angeli et al., 2000b). Clearly, it is not a restriction to assume $\gamma \in \mathcal{K}_\infty$, but it raises the question if there are possible equivalent formulations of ISS using supply pairs with $\gamma \in \mathcal{K} \setminus \mathcal{K}_\infty$ and $\alpha \notin \mathcal{K}_\infty$. This leads us to the following alternative characterization of ISS which has not previously appeared in the literature.

Proposition 2.2. *Let a system $\Sigma : \dot{x} = f(x, u)$ be given and suppose there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ and a C^1 function V satisfying (8). Assume there exist functions $\alpha, \gamma \in \mathcal{K}$ such that*

$$\langle \nabla V(x), f(x, u)\rangle \leq -\alpha(V(x)) + \gamma(\|u\|)\,.$$

If $\sup \alpha \geq \sup \gamma$ then the system Σ is ISS.

The proof of this result closely follows the lines of the result in Sontag and Teel (1995).

Proof. It suffices to consider the case when $\alpha \notin \mathcal{K}_\infty$, for otherwise Σ is ISS by definition. First let us assume that

$\sup\alpha \geq C \sup\gamma$ for some $C > 1$. Let $q \in \mathcal{K}_\infty$ be smooth and define $\rho(r) = \int_0^r q(s)ds$. Clearly $W := \rho \circ V$ is smooth, proper, and positive definite; take $\underline{\alpha}_1 = \rho \circ \underline{\alpha}$ and $\overline{\alpha}_1 = \rho \circ \overline{\alpha}$. Then $\langle \nabla W(x), f(x,u)\rangle \leq q(V(x))(-\alpha(V(x)) + \gamma(\|u\|))$.

Now either $\gamma(\|u\|) < 1/C \cdot \alpha(V(x))$ and $\langle \nabla W(x), f(x,u)\rangle \leq -(1 - 1/C)\alpha(V(x))q(V(x))$. Or otherwise $\sup\alpha > \gamma(\|u\|) \geq 1/C\cdot\alpha(V(x))$. Note that α is invertible on $[0, \sup\alpha)$. Hence with $\Theta(r) := \alpha^{-1}(C\cdot\gamma(r))$ for $r \geq 0$ we have $V(x) \leq \Theta(\|u\|)$ and therefore $\langle \nabla W(x), f(x,u)\rangle \leq -q(V(x))\alpha(V(x)) + q \circ \Theta(\|u\|)\gamma(\|u\|)$. The first term is of class $\mathcal{K}_\infty$, which gives us an ISS estimate in dissipative form.

Now assume that $\sup\alpha = \sup\gamma$. Then we can show that Σ is ISS by showing that the ISS Lyapunov implication form holds true as follows: The inverse of α exists on $[0, \sup\alpha) = [0, \sup\gamma)$, and it is easy to see that $\alpha^{-1} \circ \gamma \in \mathcal{K}_\infty$, and so is $\alpha^{-1}(\frac{1}{2}\gamma(\cdot))$. Now if $V(x) > \alpha^{-1}(\frac{1}{2}\gamma(\|u\|)) =: \widetilde{\gamma}(\|u\|)$, then $\langle \nabla V(x), f(x,u)\rangle \leq -\alpha(V(x)) + \gamma(\|u\|) \leq -\frac{1}{2}\alpha(V(x)) < 0$, which is the desired Lyapunov implication form of ISS, cf. Dashkovskiy et al. (2009b); Sontag and Wang (1995). ■

By the previous result we obtain a characterization stating when an iISS system with only positive definite decay rate is in fact ISS, which has also been observed in Ito (2006).

Corollary 2.3. *Let a system*

$$\Sigma: \ \dot{x} = f(x,u)$$

be given together with $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ *and* C^1 *Lyapunov function* V *satisfying* (8). *Assume there exist functions* $\alpha \in \mathcal{PD}$ *and* $\gamma \in \mathcal{K}$ *such that*

$$\langle \nabla V(x), f(x,u)\rangle \leq -\alpha(V(x)) + \gamma(\|u\|), \qquad (11)$$

for all $x \in \mathbb{R}^N$, $u \in \mathbb{R}^M$. *If* $\liminf_{s\to\infty}\alpha(s) \geq \sup\gamma$ *then the system* Σ *is ISS.*

Proof. Observe that any function $\alpha \in \mathcal{PD}$ for which $\liminf_{s\to\infty}\alpha(s) =: a > 0$ can be bounded from below by a $\mathcal{K}$ function β with $\sup\beta = a$: First define $b(r) = \inf_{t\geq r}\alpha(t)$; this function is continuous and increasing with $\lim_{r\to\infty} b(r) = a$ and $b(0) = 0$. Defining $\beta(r) = b(r)(1 - e^{-r})$ gives a function which in addition is strictly increasing. So we can replace α by β in (11). An application of Proposition 2.2 yields the result. ■

3. Stability of the comparison system

Consider the *comparison system* arising from (2), i.e.,

$$\dot{v} = M(v), \quad v \in \mathbb{R}^n_+, \qquad (12)$$

where M is a nonlinear operator defined by $M = -A + \Gamma$, i.e.,

$$(M(v))_i = -\alpha_i(v_i) + \sum_{j\neq i}\gamma_{ij}(v_j).$$

Throughout we assume that the functions $\alpha_i, \gamma_{ij}, \gamma_{iu}$ are locally Lipschitz, guaranteeing existence and uniqueness of solutions for (12) and also for the applicability of converse Lyapunov theorems (where locally Lipschitz right-hand sides guarantee robustness of $\mathcal{KL}$-estimates, see Teel and Praly (2000)). We denote solutions of (12) by $\phi: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, i.e., a solution of (12) at time $t \geq 0$ from an initial condition $v \in \mathbb{R}^n$ is denoted by $\phi(t,v)$.

The operator M is by definition *quasimonotone nondecreasing* (cf. Lakshmikantham, Matrosov, and Sivasundaram, 1991) which is the same as *type K* (cf. Smith, 1995), i.e., for each i, $M(v)_i \leq M(u)_i$ for any points v and u that satisfy $v \leq u$ and $v_i = u_i$. Observe that the origin is an equilibrium point of (12).

Remark 3.1. *Under the assumption that* M *is* C^1 *we have*

$$\frac{\partial M_i}{\partial v_j}(v) \geq 0, \qquad \textit{for all } i \neq j, v \in \mathbb{R}^n_+,$$

which implies that system (12) *is a* cooperative system *in the sense of Smith (1995, p.33).*

Remark 3.2 (Metzler matrices). *For the case that* M *is linear the resulting cooperative system has been widely studied in the literature. Here it can be assumed that* $M = (m_{ij})$ *is given as an* $n\times n$ *real matrix with entries satisfying* $m_{ij} \geq 0$ *whenever* $i \neq j$. *Such a matrix is called a* Metzler matrix. *We gather some well known facts from Berman and Plemmons (1979):*

The matrix M *can be written as* $M = -\alpha I + P$, *where* $\alpha \geq 0$ *is a real number,* I *is the identity matrix, and* P *is a nonnegative matrix. The origin is globally asymptotically stable with respect to the linear system* $\dot{v} = Mv, \quad v \in \mathbb{R}^n_+$, *if and only if the spectral abscissa of* M, *i.e.,*

$$a(M) := \max\{\operatorname{Re}\lambda : \lambda \textit{ is an eigenvalue of } M\},$$

is negative. An equivalent condition is to require that the spectral radius of P,

$$r(P) := \max\{|\lambda| : \lambda \textit{ is an eigenvalue of } P\},$$

satisfies $r(P) < \alpha$.

The next result is simple but vital for the applicability of results cited from the literature of monotone systems, since it allows us to consider systems evolving on $\mathbb{R}^n_+$, which is convex but not open in $\mathbb{R}^n$. When M is differentiable, that is all α_i and γ_{ij} are differentiable, we will assume one-sided limits when the derivative of M on the boundary of $\mathbb{R}^n_+$ in $\mathbb{R}^n$ is under consideration.

Lemma 3.3. *Solutions of system* (12) *starting in the positive orthant* $\mathbb{R}^n_+$ *evolve, as long as they exist, in this orthant.*

Proof. By Aubin (1991, Theorems 1.2.1 (due to Nagumo) and 1.2.3) for any $v^0 \in \mathbb{R}^n_+$ there exists a (not necessarily unique) solution to (12) confined to $\mathbb{R}^n_+$ on some interval $[0,T]$ with $T > 0$, provided that the assumptions of Nagumo's Theorem are satisfied. There are three prerequisites to check. Firstly, M has to be continuous, which it is by assumption. Secondly, $\mathbb{R}^n_+$ has to be locally compact. This is true since $\mathbb{R}^n$ is a finite dimensional vector space. Thirdly and lastly, $\mathbb{R}^n_+$ has to be a viability domain of the map M. Here it is sufficient to show that for any v on the boundary of $\mathbb{R}^n_+$, it holds for sufficiently small $h > 0$ that $v + hM(v) \in \mathbb{R}^n_+$. Now, for any v on the boundary of $\mathbb{R}^n_+$, there exists a maximal nonempty index set I, such that $v_i = 0$ for all $i \in I$. Since $M(0) = 0 \leq v$, we have, due to the type K property of M, that $M_i(v) \geq 0$ for all $i \in I$. And since for $i \notin I$, $v_i > 0$, we conclude that for sufficiently small $h > 0$, $v + hM(v) \geq 0$. This shows that Nagumo's Theorem is indeed applicable.

Extending (12) to a differential equation $\dot{v} = \overline{M}(v)$ defined on the whole of $\mathbb{R}^n$ by replacing M by the locally Lipschitz map

$$\overline{M}(v) := M(v^+)$$

with $v^+ := \max\{v, -v\}$, we obtain uniqueness of solutions due to the Picard-Lindelöf theorem. Hence every solution of (12) starting in $\mathbb{R}^n_+$ remains therein for its entire existence. ∎

An important fact regarding solutions of the comparison system (12) concerns the ordering of solutions.

Proposition 3.4 (Ordering of solutions). *Let $u^0, v^0 \in \mathbb{R}^n_+$, then on the maximal interval $J = [0, T)$ where both solutions of (12) exist, the following implications hold for $t \in J$,*

1. *if $u^0 \leq v^0$ then $\phi(t, u^0) \leq \phi(t, v^0)$;*
2. *if $u^0 < v^0$ then $\phi(t, u^0) < \phi(t, v^0)$; and*
3. *if $u^0 \ll v^0$ then $\phi(t, u^0) \ll \phi(t, v^0)$.*

The proof for right-hand sides that are only locally Lipschitz is an adaptation of the proof given in Smith (1995, Proposition 1.1, p.32).

Proof. For $k = 1, 2, \ldots$, let $\phi^k(t, x)$ denote the flow to $\dot{x} = M(x) + 1/k \cdot e$ with $e = (1, \ldots, 1)^T$. Suppose that $x_0 \leq y_0$, $t > 0$, and $\phi(t, x_0), \phi(t, y_0)$ are defined. Then $\phi^k(s, y_0 + e/k)$ is defined for all large k, say $k > K$, on $0 \leq s \leq t$, and $\phi^k(s, y_0 + e/k) \longrightarrow \phi(s, y_0)$ as $k \to \infty$, uniformly in $s \in [0, t]$ (cf. Hale (1980, Chapter 1,Lemma 3.1, p.24).

Claim: $\phi(s, x_0) \ll \phi^k(s, y_0 + e/k)$ for all $s \in [0, t]$ and $k > K$.

Proof of the claim: Fix a $k > K$, then the inequality holds by continuity for all $s > 0$ small enough. Assume the claim was false. Then there exists $t_0 : 0 < t_0 \leq t$ such that $\phi(s, x_0) \ll \phi^k(s, y_0 + e/k)$ for all $s \in [0, t_0)$ and there exists an index i such that $\phi(t_0, x_0)_i \geq \phi^k(t_0, y_0 + e/k)_i$. However, by the type K condition, as $\phi(t_0, x_0)_j \leq \phi^k(t_0, y_0 + e/k)_j$ for all $j \neq i$, we have $M_i(\phi(t_0, x_0)) \leq M_i(\phi^k(t_0, y_0 + e/k)) < M_i(\phi^k(t_0, y_0 + e/k)) + 1/k$.

The last inequality implies that for small $\varepsilon > 0$ also $M_i(\phi(t_0 - \varepsilon, x_0)) < M_i(\phi^k(t_0 - \varepsilon, y_0 + e/k)) + 1/k$ by continuity of M. Since $\phi(\cdot, x)$ and $\phi^k(\cdot, x)$ are absolutely continuous in the time variable, we have, for any such small $\varepsilon > 0$,

$$\begin{aligned}\phi(t_0, x_0)_i &= \phi(t_0 - \varepsilon, x_0)_i + \int_{t_0-\varepsilon}^{t_0} M_i(\phi(s, x_0))ds \\ &< \phi^k(t_0 - \varepsilon, y_0 + e/k)_i + \int_{t_0-\varepsilon}^{t_0} M_i(\phi^k(s, y_0 + e/k)) + \frac{1}{k}\, ds \\ &= \phi^k(t_0, y_0 + e/k),\end{aligned}$$

a contradiction, proving the claim.

If we have $x_0 < y_0$ to begin with, it follows that $\phi(t, x_0) \leq \phi(t, y_0)$ from the first part of the proof. And as solutions are unique, we must in fact have a strict inequality.

Now assume $x_0 \ll y_0$. Observe that $\phi(t, \cdot)$ is a homeomorphism that maps the order interval $[x_0, y_0]$ into $[\phi(t, x_0), \phi(t, y_0)]$ by the first part of the proof. The first order interval has nonempty interior, so must have the latter, which can only be if $\phi(t, x_0) \ll \phi(t, y_0)$. ∎

3.1. Comparison principles

The *comparison principle* (Lakshmikantham and Leela (1969a, Theorem 4.1.2, p.268) or Michel et al. (2008, Theorem 7.7.1)) states that stability properties of the trivial solution of (12) carry over to the trivial solution of system (4):

Proposition 3.5 (Comparison principle). *If the origin is globally asymptotically stable (GAS) with respect to (12), then system (4) is 0-GAS (i.e., the origin is GAS for (4) when $u \equiv 0$).*

Drawing upon essentially the same ideas, we can now state and prove a comparison principle for (integral) input-to-state stability.

Given $u \in \mathbb{R}^M$ with $M = \sum M_i$ and $w \in \mathbb{R}^n_+$, we write $G(w)$ for the vector $G(w) = \begin{bmatrix}\gamma_{1u}(w_1)\\ \vdots \\ \gamma_{nu}(w_n)\end{bmatrix}$, as well as, with slight abuse of notation, $G(u) = \begin{bmatrix}\gamma_{1u}(\|u_1\|)\\ \vdots \\ \gamma_{nu}(\|u_n\|)\end{bmatrix}$, with $u_i \in \mathbb{R}^{M_i}$ and $u = (u_1^T, \ldots, u_n^T)^T$. So the comparison system with inputs is

$$\dot{v} = M(v) + G(w)\,, \quad v, w \in \mathbb{R}^n_+\,. \tag{13}$$

Theorem 3.6 (An (i)ISS comparison principle). *Let subsystems (1) and positive definite and decrescent Lyapunov functions V_i, $i = 1, \ldots, n$, satisfying (3) as well as the dissipation estimates (2) be given. Let $M = -A + \Gamma$ and G be given by (5).*

If the comparison system (13) is ISS from w to v then system (4) is ISS from u to x; if the comparison system (13) is only integral ISS from w to v then system (4) is integral ISS from u to x.

In either case, if the smooth (integral) ISS Lyapunov function for (13) is denoted by L, then the corresponding (integral) ISS Lyapunov function for the nominal system can be taken as $V(x) = L((V_1(x_1), \ldots, V_n(x_n))^T)$.

Proof. Rewriting the prerequisites in vector notation and denoting

$$\underline{V}(x) := (V_1(x_1), \ldots, V_n(x_n))^T$$

we have along trajectories $x(t)$ of (4) for the derivative of $\underline{V}$,

$$\begin{aligned}\frac{d}{dt}\left[\underline{V}(x(t))\right] &= (\langle \nabla V_1(x_1(t)), f_1(x, u_1)\rangle, \ldots \\ &\qquad \ldots, \langle \nabla V_n(x_n(t)), f_n(x, u_n)\rangle)^T \\ &\leq M(\underline{V}(x(t))) + G(u)\,.\end{aligned}$$

By assumption there exists a smooth function $L : \mathbb{R}^n_+ \to \mathbb{R}_+$, due to an ISS (respectively, iISS) converse Lyapunov theorem, see Sontag and Wang (1995), resp. Angeli et al. (2000a), such that there exist two class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}$, so that

$$\underline{\alpha}(\|v\|) \le L(v) \le \overline{\alpha}(\|v\|)\,, \qquad \text{for all } v \in \mathbb{R}^n_+\,,$$

and there exist $\alpha \in \mathcal{PD}$ ($\alpha \in \mathcal{K}_\infty$ in the ISS case) and $\gamma \in \mathcal{K}$ such that for all $v, w \in \mathbb{R}^n_+$,

$$\langle \nabla L(v), M(v) + G(w)\rangle \le -\alpha(\|x\|) + \gamma(\|w\|)\,.$$

Now define $V(x) := L(\underline{V}(x))$. Then we have $\langle \nabla V(x), f(x,u)\rangle =$

$$\Big\langle \nabla L(\underline{V}(x)), (\langle \nabla V_1(x_1), f_1(x,u_1)\rangle, \dots, \nabla V_n(x_n), f_n(x,u_n)\rangle)^T \Big\rangle$$
$$\le \langle \nabla L(\underline{V}), M(\underline{V}(x)) + G(u)\rangle \le -\alpha(\|\underline{V}(x)\|) + \gamma(\|u\|)\,.$$

Using that $V_i(x_i) \ge \underline{\alpha}_i(\|x_i\|)$, it is clear that the last inequality implies a dissipative (integral) ISS estimate with smooth (i)ISS Lyapunov function $V = L \circ \underline{V}$. ■

The previous result might seem obvious, but it has not been formulated before in the literature. The difficulty in general will, of course, be to prove that the comparison system is iISS or ISS. A collection of sufficient conditions to deduce this will be given in the following.

At this point is useful to draw attention to the following deviation from the linear theory: Linear systems of the form $\dot{x} = Ax + Bu$, with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, map bounded inputs to bounded states if and only if A is Hurwitz. Or, equivalently, if and only if the origin is GAS for the autonomous system $\dot{x} = Ax$. In fact, the above system is ISS if and only if A is Hurwitz. One might conjecture that things are similar for general cooperative systems, but this is not necessarily so. The nonlinear (but also non-cooperative) example discussed in Sontag and Krichman (2003) illustrates that 0-GAS is strictly weaker than iISS. In particular, assumptions regarding bounds on the gradient of a Lyapunov function, as we impose in the sequel, cannot be omitted.

So a useful question to ask is: What type of stability is needed for system (12) in order to imply (integral) input-to-state stability of system (13)? The following result at least partially answers that question:

Theorem 3.7. *Assume there exist $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ and a smooth function $L : \mathbb{R}^n_+ \to \mathbb{R}_+$, such that for all $v \in \mathbb{R}^n_+$,*

$$\underline{\alpha}(\|v\|) \le L(v) \le \overline{\alpha}(\|v\|)\,, \text{ and } \langle \nabla L(v), M(v)\rangle \le -\alpha(\|v\|)\,.$$

Assume further that there exists a continuous function $q : \mathbb{R}_+ \to \mathbb{R}_+$, $q(s) > 0$ for all $s \ge 0$, satisfying $\int_0^\infty q(s)ds = \infty$ and $q(\overline{\alpha}(\|v\|)) \cdot \|\nabla L(v)\| \le 1$, for all $v \in \mathbb{R}^n_+$. Then system (13) is integral ISS. Moreover, if q can be taken to be nondecreasing, then system (13) is ISS.

Proof. Define $W(v) := \rho(L(v))$, where $\rho \in \mathcal{K}_\infty$ is defined by $\rho(r) = \int_0^r q(s)ds$. Clearly $(\rho \circ \underline{\alpha})(\|v\|) \le W(v) \le (\rho \circ \overline{\alpha})(\|v\|)$, so W is radially unbounded and decrescent. For its derivative along solutions of (13) we have $\langle \nabla W(v), M(v) + G(w)\rangle = q(L(v)) \cdot \langle \nabla L(v), M(v) + G(w)\rangle \le -\alpha(\|v\|)q(\underline{\alpha}(\|v\|)) + q(\overline{\alpha}(\|v\|))\|\nabla L(v)\|\|G(w)\| \le -\widetilde{\alpha}(\|v\|) + \|G(w)\|$. In the last inequality the function $\widetilde{\alpha}$ defined by $\widetilde{\alpha}(s) := \alpha(s)q(\underline{\alpha}(s))$ is positive definite.

If q happens to be nondecreasing we have $q(s) \ge q(0) > 0$ for all $s \ge 0$, so we may instead take $\widetilde{\alpha}(s) := q(0)\alpha(s) \le \alpha(s)q(\underline{\alpha}(s))$, which is a class $\mathcal{K}_\infty$ function.

With $\gamma(s) := \max_i \gamma_{iu}(s)$ we have $\|G(w)\| \le \gamma(\|w\|)$, so that overall we obtain $\langle \nabla W(v), M(v) + G(w)\rangle \le -\widetilde{\alpha}(\|v\|) + \gamma(\|w\|)$, which is the desired (integral) ISS estimate. ■

As an immediate consequence, by bounding the gradient of L by a constant, we obtain the following result:

Corollary 3.8. *Assume there exist $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ and a smooth function $L : \mathbb{R}^n_+ \to \mathbb{R}_+$, such that for all $v \in \mathbb{R}^n_+$,*

$$\underline{\alpha}(\|v\|) \le L(v) \le \overline{\alpha}(\|v\|)\,, \text{ and } \langle \nabla L(v), M(v)\rangle \le -\alpha(\|v\|)\,.$$

Assume further that there exists a constant $C > 0$, such that for all v, $\|\nabla L(v)\| \le C$. Then system (13) is ISS.

Proof. Apply Theorem 3.7 with $q(s) \equiv 1/C$. ■

Similarly, a weaker bound on the gradient of L along with a bound on $\overline{\alpha}$ yields only iISS:

Corollary 3.9. *Assume there exist $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ and a smooth function $L : \mathbb{R}^n_+ \to \mathbb{R}_+$, such that for all $v \in \mathbb{R}^n_+$,*

$$\underline{\alpha}(\|v\|) \le L(v) \le \overline{\alpha}(\|v\|)\,, \text{ and } \langle \nabla L(v), M(v)\rangle \le -\alpha(\|v\|)\,.$$

Assume further that there exist constants $C_1 > 0, C_2 \ge 1$, such that for all v, $\|\nabla L(v)\| \le C_1\|v\|$ and, for all $s > 0$, $C_2\overline{\alpha}(s) \ge s$. Then system (13) is iISS.

Proof. Let $q(s) = \frac{1}{C_1C_2 \cdot s}$ for $s > 1$ and $q(s) = \frac{1}{C_1C_2}$ for $s \le 1$. Observe that q is a positive and continuous function defined on $\mathbb{R}_+$. If $\|v\| > 1$ then $\|\nabla L(v)\|q(\overline{\alpha}(\|v\|)) \le \frac{C_1\|v\|}{C_1C_2\overline{\alpha}(\|v\|)} \le 1$. Otherwise, if $\|v\| \le 1$, then $\|\nabla L(v)\|q(\overline{\alpha}(\|v\|)) \le \frac{C_1\|v\|}{C_1C_2} \le 1/C_2 \le 1$.

We have $\int_0^\infty q(s)\,ds \ge \lim_{r\to\infty} \frac{1}{C_1C_2}\int_1^r 1/s\,ds = \infty$, so the function q satisfies the prerequisites of Theorem 3.7. As q is not nondecreasing, we can only deduce iISS for system (13). ■

3.2. Order and topological implications of global asymptotic stability

Scaling and growth conditions on the supply rates in Chaillet and Angeli (2008) and Ito (2006) as well as small-gain conditions in Dashkovskiy et al. (2009b) turn out to be closely connected to the concept of decay sets in (Rüffer, 2009). We define the ith *decay set* to be $\Omega_i := \{v \in \mathbb{R}^n_+ : (M(v))_i < 0\}$. This is the domain where trajectories of the comparison system (12) decrease in their ith component. Our next result states that GAS of the origin implies that nontrivial solutions are located in at least one Ω_i at any given time. Recall that if the comparison system (13) is (integral) ISS then necessarily the origin is globally asymptotically stable (GAS) with respect to the autonomous system (12).

Proposition 3.10. *If the origin is GAS with respect to (12), then the operator M satisfies*

$$M(v) \ngeq 0, \quad \forall v \in \mathbb{R}^n_+, v \neq 0 . \tag{14}$$

Proof. We argue by contradiction. Suppose there exists $v^0 > 0$, such that $M(v^0) \geq 0$. Firstly, $M(v^0) = 0$ implies the existence of an equilibrium at v^0, contradicting GAS of the origin. So we have $M(v^0) > 0$. By GAS of the origin, system (12) is forward complete. Since $M(v^0) > 0$, there exists an $\varepsilon > 0$ such that $\phi(t, v^0) > v^0$ for all $0 < t \leq \varepsilon$. Denote $v^1 := \phi(\varepsilon, v^0) > v^0$.

Now let $u^0 > v^0$. Using the ordering of solutions, Proposition 3.4, we have $\phi(t, u^0) > \phi(t, v^0) > v^0$ for all $0 < t \leq \varepsilon$. Denoting $u^1 := \phi(\varepsilon, u^0) > v^1 > v^0$, we find $\phi(t + \varepsilon, u^0) = \phi(t, u^1) > u^1 > u^0$ for $0 < t \leq \varepsilon$. Repeating the argument we obtain $\phi(t, u^0) > u^0$, $\forall t > 0$, contradicting GAS of the origin. This shows that there cannot exist $v^0 \in \mathbb{R}^n_+$, $v^0 \neq 0$, such that $M(v^0) \geq 0$. In other words, $M(v) \ngeq 0$ for all $v \in \mathbb{R}^n_+, v \neq 0$. ■

The previous result shows that—provided that the origin is GAS—every trajectory with respect to (12) has to be in one of the Ω_i sets at any given time. In other words,

$$\bigcup_{i=1}^{n} \Omega_i = \mathbb{R}^n_+ \setminus \{0\} .$$

In Dashkovskiy et al. (2007, 2009b) a condition similar to (14) has been recognized as a general small-gain type condition, guaranteeing stability of interconnections of ISS systems. For our purposes, inequality (14) can be interpreted in the following way: 'For GAS of the origin with respect to (6) the *weak small-gain condition* (14) is necessary.' In general, however, (14) alone is not a sufficient condition for GAS, as the following example illustrates.

Example 3.11. *Let a cooperative system evolving on $\mathbb{R}^2_+$ be given by*

$$\dot{v} = M(v) = \begin{bmatrix} -\dfrac{v_1}{1 + v_1^3} + v_2 \\ -v_2^4 \end{bmatrix} .$$

The operator M satisfies $M(v) \ngeq 0$ for $v \neq 0$: If $v_2 \neq 0$, then $M(v)_2 < 0$ and if $v_2 = 0$, then $M(v)_1 < 0$. Yet it can be shown that the origin is not globally asymptotically stable (e.g., the trajectory starting in $(1, 1)^T$ grows unboundedly in the v_1-direction while its v_2-component converges to zero).

Similarly to Dashkovskiy et al. (2007) we have the following result, which is based on the Knaster-Kuratowski-Mazurkiewicz (KKM) principle, cf. Knaster, Kuratowski, and Mazurkiewicz (1929); Lassonde (1990); Horvath and Lassonde (1997):

Theorem 3.12. *Assume M is such that (14) holds. Then for each $r > 0$ there exists a $v \in \mathbb{R}^n_+$, $v \gg 0$, $\|v\|_1 = r$, such that $M(v) \ll 0$. In other words, for all $r > 0$,*

$$\bigcap_{i=1}^{n} \Omega_i \cap S_r \neq \emptyset .$$

The proof of this result is essentially the same as that of the corresponding result in Dashkovskiy et al. (2007). The previous result states that there exists a *decay set*, $\Omega_{\ll} := \bigcap_{i=1}^n \Omega_i = \{v \in \mathbb{R}^n_+ : M(v) \ll 0\}$ in which solutions decrease in all components, provided that the origin is GAS. Two other decay sets of interest are $\Omega_{\leq} := \{v \in \mathbb{R}^n_+ : M(v) \leq 0\}$ and $\Omega_{<} := \{v \in \mathbb{R}^n_+ : M(v) < 0\}$. Recall that a set A is *positively invariant* if $v \in A$ implies $\phi(t, v) \in A$ for all $t \geq 0$. Assuming that M in the right-hand side of (12) is continuously differentiable, Smith (1995, Prop.2.1, p.34) proved that the sets $\Omega_{\ll}$, $\Omega_{<}$, and $\Omega_{\leq}$ are positively invariant.

We generalize this result to the case where M is only locally Lipschitz:

Lemma 3.13. *Assume that $M \ngeq 0$ then $\Omega_{\ll}$, $\Omega_{<}$, and $\Omega_{\leq}$ are invariant. Assume further that $\Omega_{\ll}$ is radially unbounded. Then the origin is globally asymptotically stable with respect to (12).*

Proof. First we show that any solution starting in $\Omega_{\ll}$ converges to the origin:

By a viability theorem (Aubin, 1991, Theorem 1.2.3) for any $u \in \mathbb{R}^n_+$ we have either that $\phi(t, u) \in \mathbb{R}^n_+$ is defined for all $t > 0$ or that $\phi(t, u) \in \mathbb{R}^n_+$ is defined only for $t \in [0, T_{\max})$ with $\lim_{t \to \infty} \|\phi(t, u)\| = \infty$, cf. the proof of Lemma 3.3.

Let $u^0 \in \Omega_{\ll}$, then there exists an $\varepsilon > 0$ such that $\phi(t, u^0) \ll u^0$ for all $t \in (0, \varepsilon]$ and $u^1 \in \Omega_{\ll}$. Define $u^1 := \phi(\varepsilon, u^0)$. We must have $\phi(t, u^1) \ll \phi(t, u^0)$ for all $t \in [0, \varepsilon]$ by Prop. 3.4 and the mentioned viability theorem. Hence we can inductively define $u^{k+1} := \phi(\varepsilon, u^k) \ll u^k$ and obtain that $\phi(t, u^0)$ is defined for all $t > 0$ and must be strictly decreasing. It follows that $\phi(t, u^0) \longrightarrow 0$, the only fixed point of M. It also follows that $\phi(t, u^0) \in \Omega_{\ll}$ for all $t > 0$, i.e., $\Omega_{\ll}$ is invariant. The same argument also works for $\Omega_{<}$ and $\Omega_{\leq}$.

For every $v^0 \in \mathbb{R}^n_+$ there exists $w \gg 0$ such that $w \geq v^0$, and $w \in \Omega_{\ll}$. By Propositions 3.4 the trajectories starting at v^0 and w are related via $0 \leq \phi(t, v^0) \leq \phi(t, w)$ for all $t \geq 0$. As we have $\phi(t, w) \to 0$, the trajectory $\phi(\cdot, v^0)$ converges to the origin. This proves that the origin is globally attractive.

Now let $\varepsilon > 0$ be given. Choose an arbitrary $r_0 \in (0, \varepsilon]$ (this step is only for compatibility with the following Lemma). Pick $w_0 \in \Omega_{\ll} \cap S_{r_0}$ and observe that it satisfies $w_0 \gg 0$. Hence we may define $\delta := \sup\{d \in \mathbb{R}_+ : \forall w \in \mathbb{R}^n_+, w \ll w_0 : \|w\| \leq d\}$. We have $\delta > 0$, and $\|w\| < \delta$ implies $w \ll w_0$. By the ordering of solutions this implies $\phi(t, w) \ll w_0$ for all $t \geq 0$, hence $\|\phi(t, w)\|_1 \leq \|w_0\|_1 = r_0 \leq \varepsilon$ for all $t \geq 0$. This proves stability. ■

Lemma 3.14. *Assume that $M \ngeq 0$ in a neighborhood of the origin. Then the origin is locally asymptotically stable with respect to (12).*

Proof. For small $r > 0$ we have due to Theorem 3.12 that $\Omega_{\ll} \cap S_r \neq \emptyset$. Hence given $\varepsilon > 0$ we may pick $r_0 \in (0, \varepsilon]$ small enough such that $\Omega_{\ll} \cap S_r \neq \emptyset$ for all $0 < r \leq r_0$. Following the second part of the proof of Lemma 3.13 we obtain stability. Local attractivity also follows as in the proof of Lemma 3.13. ■

3.3. Small-gain conditions for the comparison system

The aim of a small-gain condition is to give an algebraic criterion for global asymptotic stability of the origin. Our condition will make use of Lemma 3.13, i.e., ensure that $\Omega_{\ll}$ is radially unbounded. Note, however, that this is conservative, in the sense that it rules out certain types of subsystems in (1), see Prop. 3.20.

Theorem 3.15. *Consider the comparison system* (12) *with operators* $\Gamma, A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$. *If there exist diagonal operators*

- $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $T(v)_i = \tau_i(v_i)$, $\tau_i \in \mathcal{K}_\infty$ *satisfying* $\tau_i + \alpha_i \in \mathcal{K}_\infty$ *and*
- $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $D(v)_i = v_i + \delta_i(v_i)$, $\delta_i \in \mathcal{K}_\infty$,

such that

$$D \circ (\Gamma + T) \circ (T + A)^{-1}(v) \not\geq v, \quad \forall v > 0, \qquad (15)$$

then the origin is GAS with respect to system (12).

Note that $(T + A)$ is a diagonal operator with $\mathcal{K}_\infty$ entries on the diagonal, so its inverse is of the same shape. Moreover the composite operator $\widetilde{\Gamma} := (\Gamma + T) \circ (T + A)^{-1}$ is of the form $\widetilde{\Gamma}(v)_i = \sum_j \widetilde{\gamma}_{ij}(v_j)$, where

$$\widetilde{\gamma}_{ij} = \begin{cases} \tau_i \circ (\tau_i + \alpha_i)^{-1} & \text{if } j = i, \\ \gamma_{ij} \circ (\tau_j + \alpha_j)^{-1} & \text{otherwise,} \end{cases}$$

and $\widetilde{\gamma}_{ij}$ is of class $\mathcal{K}_\infty$ for $i = j$, and of class $\mathcal{K} \cup \{0\}$ otherwise.

Remark 3.16. *Given a locally Lipschitz continuous, positive definite function* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, *there always exists a function* $\tau \in \mathcal{K}_\infty$, *such that* $\alpha + \tau \in \mathcal{K}_\infty$. *To see this, simply note that the right-hand side derivative* $D^+\alpha(r) := \lim_{h\to 0^+} \frac{\alpha(r+h)-\alpha(r)}{h}$ *exists for all* $r \geq 0$ *and is bounded on compact intervals. So for arbitrary small* $\varepsilon > 0$ *we might take*

$$\tau(r) := \int_0^r \varepsilon + \max\{0, -D^+\alpha(s)\} ds$$

so that $D^+(\alpha + \tau)(r) > 0$ *for all* $r \geq 0$.

Thus, the hard part in applying Theorem 3.15 is to find a suitable operator D and to check the general small-gain condition, which is essentially the same task as in Dashkovskiy et al. (2007, 2009b).

Proof.[Proof of Theorem 3.15] It suffices to note that (15) or, equivalently, $D \circ \widetilde{\Gamma} \not\geq \text{id}$ implies the existence of a component-wise unbounded path σ in $\mathbb{R}^n_+$, parametrized by $\mathcal{K}_\infty$ functions σ_i such that $\widetilde{\Gamma}(\sigma(r)) \ll \sigma(r)$ for all $r > 0$, cf. Dashkovskiy et al. (2009b, Proposition 8.13). Let $\rho(r) := (T + A)^{-1}(\sigma(r))$. In particular we have for $r > 0$,

$$\begin{aligned} \widetilde{\Gamma}(\sigma(r)) = (\Gamma + T) \circ (T + A)^{-1}(\sigma(r)) \ll \sigma(r) \iff \\ (\Gamma + T)(\rho(r)) \ll (T + A)(\rho(r)) \iff \Gamma(\rho(r)) \ll A(\rho(r)) \\ \iff M(\rho(r)) = (-A + \Gamma)(\rho(r)) \ll 0 \,. \end{aligned}$$

Observe that ρ is again a strictly increasing and component-wise unbounded path in $\mathbb{R}^n_+$ parametrized by $\mathcal{K}_\infty$ functions. Furthermore, $\rho(r) \in \Omega_{\ll}$ for all $r > 0$ so that $\Omega_{\ll}$ is radially unbounded. By Lemma 3.13 it follows that the origin is GAS with respect to (12). ■

The argument can be strengthened for strongly connected networks in that we can omit the robustness term D. For technical reasons we have to assume that the network is *strongly connected via* $\mathcal{K}_\infty$ *gains*. By this we mean that Γ should be irreducible (or, Γ can be decomposed into $\Gamma = \Gamma_U + \Gamma_B$ where Γ_U consists of those γ_{ij} that are $\mathcal{K}_\infty$ and is assumed to be irreducible and Γ_B consists of those γ_{ij} that are in $\mathcal{K} \setminus \mathcal{K}_\infty$).

Theorem 3.17. *Consider the comparison system* (12) *with operators* $\Gamma, A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$. *Assume that* $\Gamma = (\gamma_{ij})$ *is irreducible and* $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ *for all i, j. If*

$$M(v) \not\geq 0, \quad \text{for all } v > 0\,, \qquad (16)$$

then the origin is GAS with respect to system (12).

Proof. By Remark 3.16 there exists an operator $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $T(v)_i = \tau_i(v_i)$, $\tau_i \in \mathcal{K}_\infty$ satisfying $\tau_i + \alpha_i \in \mathcal{K}_\infty$. So we can rewrite (16) as $M = -A + \Gamma \not\geq 0 \iff (\Gamma + T) \circ (T + A)^{-1} =: \widetilde{\Gamma} \not\geq \text{id}$. The operator $\widetilde{\Gamma}$ is again of the form $\widetilde{\Gamma} = (\widetilde{\gamma}_{ij})$. We have $\widetilde{\gamma}_{ij} \in \mathcal{K}_\infty \cup \{0\}$ and $\widetilde{\Gamma}$ is irreducible. Now by Rüffer (2009, Theorem 5.4) there exists a component-wise unbounded path σ in $\mathbb{R}^n_+$, parametrized by $\mathcal{K}_\infty$ functions σ_i such that $\widetilde{\Gamma}(\sigma(r)) \ll \sigma(r)$ for all $r > 0$.

Similarly as in the proof of Theorem 3.15 it follows that there exists a path ρ with strictly increasing and unbounded component functions, such that $M(\rho(r)) \ll 0$ for all $r > 0$, cf. Dashkovskiy et al. (2009b, Theorem 8.11) or Rüffer (2009, Theorem 5.5). Again we conclude using Lemma 3.13. ■

As an immediate consequence of the preceding results, we have:

Corollary 3.18. *Consider system* (4) *decomposed into subsystems* (1). *Assume for each subsystem* (1) *there exists a Lyapunov function* V_i *satisfying* (3) *as well as the dissipation inequality* (2). *Let* $\Gamma, A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *given by* (5) *and let* $M = -A + \Gamma$. *Assume that either*

1. *there exist diagonal operators*
   (a) $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $T(v)_i = \tau_i(v_i)$, $\tau_i \in \mathcal{K}_\infty$ *satisfying* $\tau_i + \alpha_i \in \mathcal{K}_\infty$ *and*
   (b) $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $D(v)_i = v_i + \delta_i(v_i)$, $\delta_i \in \mathcal{K}_\infty$,

   such that (15) *holds for all* $v > 0$; *or that*
2. $\Gamma = (\gamma_{ij})$ *is irreducible with* $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ *for all i, j, and that* (16) *holds for all* $v > 0$.

Then system (4) *is 0-GAS (i.e., the origin is GAS for* (4) *when* $u \equiv 0$*).*

The above small-gain conditions are based on Lemma 3.13, which requires a *radially* unbounded set $\Omega_{\ll}$. We will see in the following example that the origin can be globally asymptotically stable although $\Omega_{\ll}$ is not radially unbounded (though it is still at least unbounded in one coordinate direction).

Example 3.19. *Let $\alpha_1(s) = s/(1+s)$, which is of class $\mathcal{K} \setminus \mathcal{K}_\infty$ with $\lim_{s\to\infty} \alpha_1(s) = 1$. Let $\gamma_{12} \in \mathcal{K}_\infty$, $\alpha_2 \in \mathcal{PD}$ and consider the system*

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -\alpha_1(v_1) + \gamma_{12}(v_2) \\ -\alpha_2(v_2) \end{bmatrix} =: M(v)\,,$$

which may be interpreted as a comparison system of a cascade of a GAS system driving an iISS system. By considering the cases $v_2 = 0$ and $v_2 > 0$ it is clear that $M \ngeq 0$.

The origin is GAS for the v_2-subsystem, and clearly for small $\varepsilon > 0$ the compact set $A_\varepsilon = \{v_2 \in \mathbb{R}_+ : v_2 \leq \gamma_{12}^{-1}(1) - \varepsilon\}$ will be reached by any trajectory in finite time. The set Ω_2 is the whole of $\mathbb{R}^2_+$ without the v_2-axis.

The set Ω_1 is given by $\Omega_1 = \{v \in \mathbb{R}^2_+ : v_2 < \gamma_{12}^{-1}(\alpha_1(v_1))\}$, i.e., the region below the graph of $\gamma_{12}^{-1} \circ \alpha_1 \in \mathcal{K} \setminus \mathcal{K}_\infty$, cf. Figure 1. Now using a domination argument as in Lemma 3.13 we prove

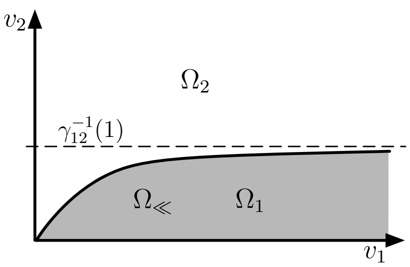

Figure 1: The sets $\Omega_1, \Omega_2, \Omega_\ll$ in Example 3.19.

that the origin is GAS for the composite system: Fix some small $\varepsilon > 0$. Any trajectory will eventually enter A_ε, by Lemma 3.13 it will then be dominated by a trajectory starting in $\Omega_\ll = \Omega_1 \cap \Omega_2$. All trajectories in $\Omega_\ll$ approach the origin, so this proves global attractivity, and from Lemma 3.14 we have local stability.

Clearly the small-gain type conditions impose some restrictions on the type of system under consideration. In fact, this restriction affects the functions α_i which constitute the operator A, as we shall see next. In light of Proposition 2.2 this means that some of the subsystems in the original interconnection satisfy stronger stability properties than just iISS.

Proposition 3.20. *Consider the comparison system (12) with operators $\Gamma, A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $A = \operatorname{diag}(\alpha_i)$. If there exists a path $\sigma \in \mathcal{K}^n_\infty$ such that $M = -A + \Gamma$ satisfies*

$$M(\sigma(r)) \ll 0, \qquad \text{for all } r > 0\,, \tag{17}$$

then for all i such that there exists j with $\gamma_{ij} \neq 0$, the corresponding function α_i is bounded from below by a function of the same class as γ_{ij}. In particular, if $\gamma_{ij} \in \mathcal{K}_\infty$ then α_i is bounded from below by (and hence can be assumed to be) a class $\mathcal{K}_\infty$ function and if $\gamma_{ij} \in \mathcal{K} \setminus \mathcal{K}_\infty$ then at least $\liminf_{s\to\infty} \alpha_i(s) > 0$.

Proof. By (17) we have $\alpha_i(\sigma_i(r)) > \sum_{k\neq i} \gamma_{ik}(\sigma_k(r)) \geq \gamma_{ij}(\sigma_j(r))$ for any j, all $r > 0$. Hence $\alpha_i > \gamma_{ij} \circ \sigma_j \circ \sigma_i^{-1}$, where $\gamma_{ij} \circ \sigma_j \circ \sigma_i^{-1}$ is a function of the same class as γ_{ij}. From here the claim follows. ∎

3.4. A remark on the construction of Lyapunov functions

Despite the restriction imposed by the small-gain condition (as it implies that some subsystems have to be "more stable" in the sense of Proposition 3.20), the small-gain condition is quite useful, as it allows the construction of at least a non-smooth Lyapunov function for the composite system using the approach detailed in Dashkovskiy et al. (2009b). To treat this case thoroughly, care has to be taken when the derivative of the locally Lipschitz continuous Lyapunov function is considered at points where it is not differentiable in the classical sense. For these details the reader is referred to Dashkovskiy et al. (2009b) where this issue has been dealt with using Clarke's generalized derivatives. Here we only sketch the procedure, assuming that derivatives exist where we take them. See also Dashkovskiy et al. (2009a) for an alternative approach.

Suppose there exists a $\mathcal{K}_\infty$-path $\rho : \mathbb{R}_+ \to \mathbb{R}^n_+$, such that

$$M(\rho(r)) = -A(\rho(r)) + \Gamma(\rho(r)) \ll 0\,, \qquad \text{for all } r > 0\,.$$

Suppose further that the functions α_i constituting A are of class $\mathcal{K}_\infty$ and hence invertible (with inverses again of class $\mathcal{K}_\infty$).

We know that $\langle \nabla V_i(x_i), f_i(x, u_i)\rangle \leq -\alpha_i(V_i(x_i)) + \sum_{j\neq i} \gamma_{ij}(V_j(x_j))$, for all $i = 1, \ldots, n$. If the vector $\underline{V}(x) := (V_1(x_1), \ldots, V_n(x_n))^T$ is in Ω_i for one particular i, then we have $\langle \nabla V_i(x_i), f_i(x, u_i)\rangle \leq -\alpha_i(V_i(x_i)) + \sum_{j\neq i} \gamma_{ij}(V_j(x_j)) < 0$. In other words, the following implication holds:

$$V_i(x_i) \geq \alpha_i^{-1}\Big(\sum_{j\neq i} \gamma_{ij}(V_j(x_j))\Big) \implies \dot{V}_i < 0\,.$$

Now define a function $V(x) = \max_i \rho_i^{-1}(V_i(x_i))$. It is straight forward to check that V satisfies an estimate of the form (8).

Assume that for a given x we have $V(x) = \rho_i^{-1}(V_i(x_i))$ for a particular i. Then it follows that $V_j(x_j) \leq \rho_j(V(x))$ for all j. So we have $\alpha_i^{-1}\big(\sum_{j\neq i} \gamma_{ij}(V_j(x_j))\big) \leq \alpha_i^{-1}\big(\sum_{j\neq i} \gamma_{ij}(\rho_j(V(x)))\big) < \rho_i(V(x)) = V_i(x_i)$, and hence $\langle \nabla V(x), f(x,u)\rangle = \underbrace{(\rho_i^{-1})'(V_i(x_i))}_{>0} \cdot \underbrace{\langle \nabla V_i(x_i), f_i(x, u_i)\rangle}_{<0} < 0$, at least for all points of differentiability of ρ_i^{-1}, which can be assumed to be almost everywhere.

The locally Lipschitz continuous Lyapunov function that we have just constructed only serves to show GAS of the origin for the composite system (4). However, if the sums of gains are extended by an external input, a slightly modified procedure still works, leading to an ISS Lyapunov function for the composite system (Dashkovskiy et al., 2009b,a).

4. Conclusions

In this work we have established stability criteria in terms of Lyapunov functions for cooperative systems arising as comparison systems of large-scale interconnections of (integral) ISS systems. Using a comparison theorem which says that the nominal system satisfies essentially the same types of stability properties as the comparison system, we have provided several results for stability of nonlinear large-scale systems.

Based on the geometric implications of global asymptotic stability of the origin with respect to the comparison system, we have derived a small-gain-type condition for stability. Furthermore, we have shown that this condition itself restricts the class of integral ISS systems that can be interconnected.

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