Convergent Systems vs. Incremental Stability

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Abstract

Two similar stability notions are considered; one is the long established notion of convergent systems, the other is the younger notion of incremental stability. Both notions require that any two solutions of a system converge to each other. Yet these stability concepts are different, in the sense that none implies the other, as is shown in this paper using two examples. It is shown under what additional assumptions one property indeed implies the other. Furthermore, this paper contains necessary and sufficient characterizations of both properties in terms of Lyapunov functions.

Key words: convergent systems, incremental stability, time-varying nonlinear systems, converse Lyapunov result

1. Introduction

In this paper we study and compare two long established and related stability notions, namely those of incremental stability [24, 10, 2, 26] and convergence [7, 23, 13]. These stability notions have received an increased interest in recent years due to their potential application in synchronisation [17, 5, 21], nonlinear output regulation [15], steady-state analysis of nonlinear systems [12] and many other nonlinear control problems. We refrain from giving a further and exhaustive overview on these, and related, stability notions; rather, we study and compare in detail the notions of incremental stability as defined in [2] and convergent systems as defined in [13]. The reason for this study is that, although these stability notions appear to be similar, they are in fact different. On the one hand, we will make explicit these differences and, on the other hand, we will present conditions under which one stability property implies the other.

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Let us introduce the definitions of convergence and incremental stability. Consider hereto a system

$$\dot{x}(t) = f(t, x) \tag{1}$$

with $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ measurable in t and locally Lipschitz in $x \in \mathbb{R}^n$, uniformly for t in compact sets (this assumption guarantees uniqueness and local existence of solutions, cf. [18]). We say that a set $A \subset \mathbb{R}^n$ is *positively invariant* under (1) if $x^0 \in A$ implies that for all $t^0 \in \mathbb{R}$, $x(t, t^0, x^0) \in A$ for all $t \geqslant t^0$.

Let $\mathcal{X} \subset \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. We are interested in two stability concepts, defined as follows.

Definition 1 (cf. [15, 13]). *System* (1) *is* uniformly convergent in a positively invariant set $\mathcal{X}$ *if*

1. *all solutions $x(t, t^0, x^0)$ exist for all $t \geqslant t^0$ for all initial conditions $(t^0, x^0) \in \mathbb{R} \times \mathcal{X}$;*
2. *there exists a unique solution $\overline{x}(t)$ in $\mathcal{X}$ defined and bounded for all $t \in \mathbb{R}$;*
3. *the solution $\overline{x}(t)$ is uniformly[1] asymptotically stable in $\mathcal{X}$, i.e., there exists a function $\beta \in \mathcal{KL}$ such that for all $(t^0, x^0) \in \mathbb{R} \times \mathcal{X}$ and $t \geqslant t^0$,*

   $\|x(t, t^0, x^0) - \overline{x}(t)\| \leqslant \beta\big(\|x^0 - \overline{x}(t^0)\|, t - t^0\big).$

System (1) *is* globally uniformly convergent *if it is uniformly convergent in* $\mathbb{R}^n$*.*

For a uniformly convergent system, the unique, bounded uniformly asymptotically stable solution $\overline{x}(t)$ is called a *steady-state solution*.

Definition 2 (cf. [2]). *System* (1) *is* incrementally asymptotically stable *(IS for short)* in a positively invariant set $\mathcal{X} \subset \mathbb{R}^n$ *if there exists a function $\beta \in \mathcal{KL}$ such that for any $\xi^1, \xi^2 \in \mathcal{X}$ and $t \geqslant t^0$,*

$$\|x(t, t^0, \xi^1) - x(t, t^0, \xi^2)\| \leqslant \beta(\|\xi^1 - \xi^2\|, t - t^0)\,. \tag{2}$$

In the case $\mathcal{X} = \mathbb{R}^n$ we say that system (1) *is* globally incrementally stable *(GIS), or just* incrementally stable.

The definitions given here are for seemingly very general time-varying systems. Still, implicit to both definitions is that solutions to (1) with initial conditions in $\mathcal{X}$ exist for all forward times. Also note that in contrast to the definition given here, most existing notions of incremental stability, e.g. [2], are defined only for systems with right-hand sides not explicitly depending on time. Furthermore, item 1 in Definition 1 is actually redundant, since in this paper we define convergence with respect to a positively invariant set. However, historically convergence would be defined using item 1 instead of the positively invariant set.

As argued above the properties of incremental stability and convergence are very useful in tackling a range of nonlinear control problems. Moreover, since

[1] In Definition 1 the uniqueness of the solution $\overline{x}(t)$ is in fact a consequence of its *uniform* asymptotic stability, cf. [15, p.15, Property 2.15].

the definition of uniform convergence implies the existence of a unique bounded (uniformly globally asymptotically stable) solution, termed the steady-state solution, the convergence property is a powerful tool for steady-state (performance) analysis of nonlinear (control) systems. We note that the existence of such a well-defined steady-state solution is not implied by the incremental stability property.

Both the incremental stability and the uniform convergence property can be thought of as an open-loop observability property, i.e., the possibility to construct an observer for the system that is based entirely on past input data.

In [2], equivalent notions of incremental stability have been derived, most notably among them a characterization in terms of a merely continuous Lyapunov function, albeit only for systems with right-hand sides not depending explicitly on time. Other notions of incremental stability as, e.g., in [25] are invariant under changes of coordinates. Here we focus on a notion similar to that in [2], and, by extending a result from [2], we present a Lyapunov characterization of incremental stability (see Theorem 5) for systems with right-hand sides depending explicitly on time. In contrast, to date and to the best of our knowledge no necessary *and* sufficient characterization in terms of a Lyapunov function is known for the convergence property; however, a number of sufficient conditions for uniform convergence based on Lyapunov functions can be found in [16, 7, 23, 15, 14]. In addition, we also provide a characterization of global uniform convergence in terms of a smooth Lyapunov function.

Another difference between the two properties is that incremental stability, as defined in [2], is not invariant under changes of coordinates. For the purposes of this paper, however, we will not pursue this aspect further and instead refer the interested reader to the discussion in [26].

On the one hand, it might seem obvious that in general incremental stability does not imply convergence, cf. Example 4 in this paper. Namely, for systems whose trajectories converge to each other and at the same time tend to infinity together, clearly, the unique $\overline{x}(t)$ as in Definition 1, if it exists, would not be bounded. On the other hand, one might be led to believe that the converse implication could be true, i.e., that a convergent system is incrementally stable, since when two different trajectories $x(t, t^0, \xi^1)$ and $x(t, t^0, \xi^2)$ tend to $\overline{x}(t)$, then obviously they also tend to each other, as is depicted in Figure 1. This would imply that the class of convergent systems is a proper subset of the class of incrementally stable systems.

In this paper, we will argue that incremental stability and convergence are indeed distinct stability notions. This claim is supported by several examples, presented in Section 2. Herein, we first show that convergence does not imply incremental stability, since the convergence of two trajectories towards each other does not have to be uniform in the distance of the initial conditions. Second, we show that if any two trajectories become eventually close (as is the case in incrementally stable systems), that does not imply the existence of a solution that is bounded forward and backward in time (as in convergent systems). Still, these stability notions are related and we will present sufficient conditions in Section 3 under which the one property implies the other. In that section we also provide converse Lyapunov results for incrementally stable and uniformly convergent systems, which are of independent interest. All proofs of these main results are provided in an appendix. The paper will close with conclusions in Section 4.

Notation: By $\mathbb{R}_+$ we denote the real half line $[0,\infty)$. Throughout the paper we will denote by $\mathcal{K}$ the class of continuous and strictly increasing functions $\kappa\colon \mathbb{R}_+ \to \mathbb{R}_+$ for which $\kappa(0) = 0$. A function ρ is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A continuous function $\beta\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if for any fixed $s \geqslant 0$, $\beta(\cdot, s) \in \mathcal{K}$ and $\beta(s,\cdot)$ is non-increasing with $\lim_{t\to\infty} \beta(s,t) = 0$.

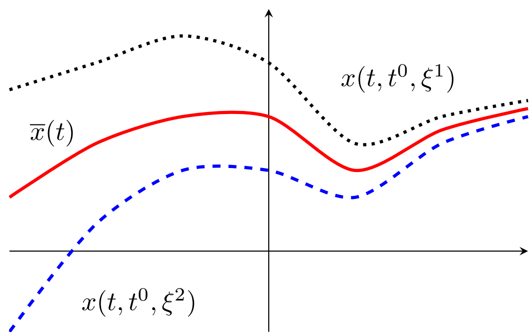

Figure 1: The uniform convergence property: Two solutions tending to the unique bounded solution $\overline{x}(\cdot)$.

2. Examples

Our first example is a system whose trajectories spiral counter-clockwise towards a bounded solution $\overline{z}(t)$, but the further away from $\overline{z}(t)$ one starts, the faster the angular velocity is. So the solution $\overline{z}(t)$ is globally asymptotically stable, which is shown using a quadratic Lyapunov function, while two solutions starting at $t = 0$ an appropriately chosen distance $\epsilon > 0$ away from each other get separated arbitrarily much in finite time, if they both start far away from $\overline{z}(t)$.

Example 3 (A uniformly convergent system that is not GIS)**.** *For $z \in \mathbb{R}^2$ consider the system*

$$\begin{aligned}\dot{z}(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + \|z(t) - \overline{z}(t)\|_2^2 \begin{pmatrix} -z_2(t) + \sin(t) \\ z_1(t) - \cos(t) \end{pmatrix} \\ - \operatorname{sat}_1\left(\|z(t) - \overline{z}(t)\|_2^2\right) \begin{pmatrix} z_1(t) - \cos(t) \\ z_2(t) - \sin(t) \end{pmatrix},\end{aligned} \tag{3}$$

where $\overline{z}(t) = (\cos t, \sin t)^\top$ *and* $\operatorname{sat}_r : \mathbb{R} \to \mathbb{R}$ *is given by*

$$\operatorname{sat}_r(s) = \begin{cases} -r & \text{if } s \leqslant -r \\ s & \text{if } |s| < r \\ r & \text{if } s \geqslant r. \end{cases}$$

Obviously, $\overline{z}(t)$ *is a bounded solution of* (3) *on* $\mathbb{R}$*. Now consider the time-varying quadratic Lyapunov function* $V(t,z) = \frac{1}{2}\|z - \overline{z}(t)\|_2^2$*. Then it can be verified that*

$$\begin{aligned} \dot{V} &= \frac{d}{dt} V(t, z(t)) \\ &= -\operatorname{sat}_1\left(\|z(t) - \overline{z}(t)\|_2^2\right)\|z(t) - \overline{z}(t)\|_2^2 < 0 \end{aligned}$$

whenever $z(t) \neq \overline{z}(t)$*, proving uniform global asymptotic stability of the bounded solution* $\overline{z}(t)$ *of* (3)*. Hence the system is globally uniformly convergent. (See also Theorem 7.) Rewriting* $x(t) := z(t) - \overline{z}(t)$ *in polar coordinates* (r, ϕ) *yields, in the region where* $r > 1$*,*

$$\begin{aligned} \dot{r} &= -r \\ \dot{\phi} &= r^2\,, \end{aligned}$$

which has solutions for initial values (in polar coordinates) $(r^0, \phi^0)^\top$*,* $r^0 > 1$*, explicitly given by*

$$\begin{aligned} r(t) &= r^0 e^{-t} \\ \phi(t) &= \phi^0 + \frac{(r^0)^2}{2}\left(1 - e^{-2t}\right), \end{aligned} \tag{4}$$

for $t \geqslant 0$ *such that* $r(t) > 1$*.*

Claim: *With* $M = \frac{2\pi e}{e-1}$ *there exist points* ξ^1, ξ^2 *with* $\|\xi^1 - \xi^2\| \leqslant M$ *such that for any* $R > 1$ *sufficiently large, cf.*

$$\|z(1/2, 0, \xi^1) - z(1/2, 0, \xi^2)\| = \frac{\sqrt{R+M} + \sqrt{R}}{\sqrt{e}}\,,$$

see Fig. 2. This implies that there cannot exist a $\mathcal{KL}$ *function* β *such that* (2) *holds and hence the system is not GIS.*

Proof of the claim. Let $R > 1$ be large enough such that solutions $z(t, 0, \xi^i)$ starting in $\xi^1 = \left(\sqrt{R+M}, 0\right)^\top + \overline{z}(0)$ and $\xi^2 = \left(\sqrt{R}, 0\right)^\top + \overline{z}(0)$ satisfy $\|z(t,0,\xi^i) - \overline{z}(t)\| = \|x(t, 0, \xi^i - \overline{z}(0))\| > 1$ for all $t \in [0, 1/2]$, $i = 1, 2$. Observe that $\|\xi^1 - \xi^2\| = \left(M + \sqrt{R}\left(2\sqrt{R} - 2\sqrt{R+M}\right)\right)^{1/2} \leqslant \sqrt{M}$. Using (4), at time $t = 1/2$ the difference of the respective angle functions $\phi_i(t) = \phi(t, 0, \xi^i - \overline{z}(0))$, $i = 1, 2$, satisfies

$$\phi_1(1/2) - \phi_2(1/2) = (R+M)/2(1 - e^{-2t}) - R/2(1 - e^{-2t}) = \frac{M}{2}(1 - 1/e) = \pi\,. \tag{5}$$

Denote correspondingly $r_i(t) = r(t, 0, \xi^i - \overline{z}(0))$, $i = 1, 2$. Using (5),

$$\|z(1/2, 0, \xi^1) - z(1/2, 0, \xi^2)\|$$

$$= \|x(1/2, 0, \xi^1 - \overline{z}(0)) - x(1/2, 0, \xi^2 - \overline{z}(0))\|$$
$$= r_1(1/2) + r_2(1/2)$$
$$= \sqrt{R+M}e^{-1/2} + \sqrt{R}e^{-1/2} = \frac{\sqrt{R+M}+\sqrt{R}}{\sqrt{e}},$$

where the second equality is owed to the fact that $x(1/2, 0, \xi^1)$ and $x(1/2, 0, \xi^2)$ are vectors pointing in opposite directions, as certified by (5). □

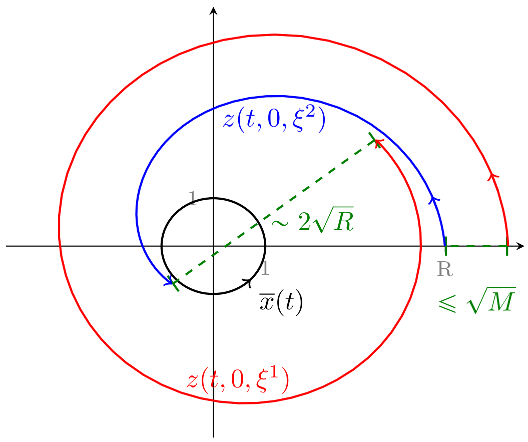

Figure 2: The two trajectories in Example 3 start on the positive real half line with an initial separation less than $\sqrt{M}$ at time $t = 0$ and the lesser initial distance to the origin is R. At time $t = 1/2$ and under a suitable time-varying change of coordinates, the arguments of the trajectories are shifted by 180° so that the separation distance is about $2\sqrt{R}$.

In the previous example, we have in fact shown that a bounded trajectory can be globally asymptotically stable (GAS) and trajectories are not GAS with respect to each other.

Another example is the system

$$\dot{x} = -\operatorname{sat}_1 x.$$

Here the bounded solution is the origin $\overline{x}(t) \equiv 0$. The origin is globally asymptotically stable (hence the system is convergent), and yet the difference between trajectories starting out arbitrarily close remains constant before the first of them enters the unit ball. This system could be considered *marginally GIS*, as the distance between trajectories cannot increase arbitrarily much in finite time as in Example 3.

The second type of example, discussed next and concerning a GIS system that is not uniformly convergent, is much easier to construct than the first, as we only have to construct a system with one globally uniformly asymptotically stable solution, which is unbounded in forward time. In fact, even a one-dimensional counterexample can be realized.

Example 4 (A system that is GIS but not uniformly convergent)**.** *Consider*

$$\dot{x}(t) = t - x\,, \quad x \in \mathbb{R}\,, \tag{6}$$

which has the explicit solution

$$\begin{aligned} x(t, t^0, x^0) &= x^0 e^{-t+t^0} + \int_{t^0}^{t} e^{s-t} s ds \\ &= x^0 e^{-t+t^0} + \big[(s-1)e^{s-t}\big]_{s=t^0}^{s=t} \\ &= x^0 e^{-t+t^0} + (t-1) - (t^0-1)e^{t^0-t}\,. \end{aligned}$$

Obviously, the solution passing through $x^0 = 0$ at $t^0 = 0$ is unbounded. Hence the system cannot be globally convergent (since otherwise the same solution would have to be attracted to a bounded solution as $t \to \infty$).

Taking any $\xi^1, \xi^2 \in \mathbb{R}$ then

$$\frac{d}{dt}\big[x(t, t^0, \xi^1) - x(t, t^0, \xi^2)\big] = -\big(x(t, t^0, \xi^1) - x(t, t^0, \xi^2)\big)\,,$$

which implies

$$\|x(t, t^0, \xi^1) - x(t, t^0, \xi^2)\| \leqslant \|\xi^1 - \xi^2\| e^{-t},$$

which, in turn, represents a $\mathcal{KL}$-estimate on the difference between any two solutions. So the system (6) *is GIS.*

This in turn implies that the solution passing through $x^0 = 0$ is globally attractive, and hence no bounded solution can exist, so the system cannot be convergent on a subset of $\mathbb{R}^n$.

On the one hand, the above examples clearly show that the stability notions of convergence and incremental stability are different. On the other hand, the classes of GIS and convergent systems also have nonempty intersection: for example, any linear system $\dot{x} = Ax$ with A Hurwitz satisfies both properties.

3. When does uniform convergence imply incremental stability and vice versa?

In this section, we present several sufficiency results regarding convergence and incremental stability that show under which conditions one property implies the other (see Sections 3.2 and 3.3).

In order to obtain one of the main results, we require a Lyapunov characterization for GIS for systems of the form (1), which is of independent interest. This converse Lyapunov result is presented in Section 3.1. Here we also provide a Lyapunov characterization for global uniform convergence, which is essentially based on standard converse Lyapunov results for uniform asymptotic stability. However, we note that such a full Lyapunov-based characterisation of convergence was lacking in the literature.

Our main results aim at answering the following two questions:

1. When is a uniformly convergent system IS?
2. When is an IS system uniformly convergent?

The first question will be answered in Section 3.2 and the second one in Section 3.3.

Briefly, Theorems 8 and 11 show that incremental stability and uniform convergence are in fact equivalent, when system (1) evolves in a compact set.

On a global scale, more restrictive and less symmetric assumptions have to be added, and we present one main theorem for each direction (Theorems 10 and 12).

All proofs in this section are provided in the appendix.

3.1. Converse Lyapunov results

In [2] a characterization of GIS in terms of a merely continuous Lyapunov function has been derived for systems of the form

$$\dot{x} = f(x, d), \tag{7}$$

where d is an arbitrary, measurable disturbance function taking values in a closed subset $\mathcal{D}$ of $\mathbb{R}^m$. However, the formulation (7) does not encode an explicit dependence of the right-hand side f on time, and subsequently the Lyapunov function shown to exist in [2] does not depend on time either.

Similarly, the existence result of a smooth Lyapunov function from a $\mathcal{KL}$-estimate in [22], while capable of capturing time-varying systems through the state-space augmentation

$$\dot{\xi} = \frac{d}{dt}\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} f(t,x) \\ 1 \end{pmatrix} =: F(\xi),$$

imposes stronger conditions on the time-dependence than necessary for existence and uniqueness of solutions, which we seek to avoid here.

A recent converse result established in [8] provides locally Lipschitz continuous Lyapunov functions for non-autonomous differential equations, utilizing the theory of skew-product flows, which adds more technical overhead than would seem appropriate for our purposes.

Therefore, we propose the following result that shows the existence of a *time-varying* Lyapunov function for global incremental stability.

Theorem 5. *System* (1) *is GIS if and only if there exist a continuous function* $W\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$*, functions* α_1*,* α_2*,* α_3 *of class* $\mathcal{K}_\infty$ *such that*

1. *the inequalities*

$$\alpha_1(\|x^1 - x^2\|) \leqslant W(t, x^1, x^2) \leqslant \alpha_2(\|x^1 - x^2\|) \tag{8}$$

hold for all x^1*,* $x^2 \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$*;*

2. *along trajectories of* (1) *for any* ξ^1, $\xi^2 \in \mathbb{R}^n$, *and any* $t \geqslant t^0$ *it holds that*

$$W\big(t, x(t,t^0,\xi^1), x(t,t^0,\xi^2)\big) - W(t^0,\xi^1,\xi^2) \leqslant -\int_{t^0}^{t} \alpha_3\big(\|x(\tau,t^0,\xi^1) - x(\tau,t^0,\xi^2)\|\big) d\tau. \tag{9}$$

In this result we can trade the unboundedness of α_3 for a Lipschitz-like property of the Lyapunov function W as formalized in the next corollary. This corollary will be instrumental in the proof of Theorem 12.

Corollary 6. *If system* (1) *is GIS then there exist a continuous function* $W\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, *functions* α_1, $\alpha_2 \in \mathcal{K}_\infty$, *and a positive definite function* α_3 *such that the inequalities* (8) *and* (9) *hold. Moreover, there exists a function* $\gamma \in \mathcal{K}_\infty$ *so that for all* $z^1, z^2 \in \mathbb{R}^n \times \mathbb{R}^n$ *and all* t^0,

$$|W(t^0, z^1) - W(t^0, z^2)| \leqslant \gamma(\|z^1 - z^2\|). \tag{10}$$

Condition (10) implies uniform continuity of W with respect to z, which itself is equivalent to the existence of a class $\mathcal{K}$ function ζ such that

$$\zeta(|W(t^0, z^1) - W(t^0, z^2)|) \leqslant \|z^1 - z^2\|$$

for all $t^0 \in \mathbb{R}$ and all $z^1, z^2 \in \mathbb{R}^n \times \mathbb{R}^n$. However, ζ does not necessarily need to be invertible, and hence (10) is a bit stronger than uniform continuity.

The proof of the preceding theorem is rather complex, see Appendix A.1. In contrast, for global uniform convergence we can obtain a corresponding characterization using a standard converse Lyapunov result, [11, Theorem 23], which for our purposes reads as follows.

Theorem 7. *Assume that system* (1) *is globally uniformly convergent. Assume that the function* f *is continuous in* (t,x) *and* $\mathcal{C}^1$ *with respect to the* x *variable. Assume also that the Jacobian* $\frac{\partial}{\partial x} f(t,x)$ *is bounded, uniformly in* t. *Then there exists a* $\mathcal{C}^1$ *function* $V\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$, *functions* α_1, α_2, *and* $\alpha_3 \in \mathcal{K}_\infty$, *and a constant* $c \geqslant 0$ *such that*

$$\alpha_1(\|x - \overline{x}(t)\|) \leqslant V(t,x) \leqslant \alpha_2(\|x - \overline{x}(t)\|) \tag{11}$$

and

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leqslant -\alpha_3(\|x - \overline{x}(t)\|) \tag{12}$$

and

$$V(t,0) \leqslant c, \quad t \in \mathbb{R}. \tag{13}$$

Conversely, if a differentiable function $V\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}_+$ *and functions* $\alpha_i \in \mathcal{K}_\infty$, $i = 1,2,3$, *and* $c \geqslant 0$ *are given such that for some trajectory* $\overline{x}\colon \mathbb{R} \to \mathbb{R}^n$ *estimates* (11)–(13) *hold, then system* (1) *must be globally uniformly convergent and the solution* $\overline{x}$ *is the unique bounded solution as in Definition 1.*

The proof of this result is an application of Massera's result to (1) under the change of coordinates $z(t) = x(t) - \overline{x}(t)$. The only addition is (13), which is equivalent to the boundedness of $\overline{x}(t)$. For if $\overline{x}(t)$ is bounded forward and backward in time, i.e., $\sup_{t\in\mathbb{R}} \|\overline{x}(t)\| < \infty$, then $0 \leqslant V(t,0) \leqslant \alpha_2(\|\overline{x}(t)\|) \leqslant \sup_{\tau\in\mathbb{R}} \alpha_2(\|\overline{x}(\tau)\|) =: c < \infty$. On the other hand, if (13) holds, then the solution $\overline{x}$ must be bounded forward and backward in time, since for all $t \in \mathbb{R}$ we have

$$\|\overline{x}(t)\| \leqslant \alpha_1^{-1}(V(t,0)) \leqslant \alpha_1^{-1}(c) < \infty.$$

3.2. From convergence to incremental stability

The following theorem is a new sufficiency condition for incremental stability.

Theorem 8. *Suppose system* (1) *is uniformly convergent on a compact set $\mathcal{X}$. Then, it is also incrementally stable on that set.*

Remark 9. *Let us now briefly revisit Example 3 given the result in Theorem 8. Example 3 concerns a system that is globally uniformly convergent, but not GIS. Since the system is globally uniformly convergent, it is also uniformly convergent on compact, positively invariant sets and Theorem 8 shows that it is also incrementally stable on such compact sets. Note that the argument against it being GIS does not imply that it is not incrementally stable on compact positively invariant sets, since R in the example can not be chosen arbitrarily large when considering initial conditions on compact sets.*

If system (1) does not evolve in a compact set then additional conditions on the vector field f allow to infer one stability property from the other.

Let us now formulate conditions under which a *globally* convergent system is also *globally* IS. In general, while also for convergent systems all trajectories approach each other, they may do so non-uniformly in the initial separation distance, as could be seen from Example 3. The idea of the next result is to enforce this uniformity by an additional assumption on a (non-strict, quadratic) Lyapunov function for a globally convergent system.

Theorem 10. *Suppose system* (1) *is globally uniformly convergent. Assume that also the assumptions of Theorem 7 are satisfied. Assume further that there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$, i.e. $P = P^\top > 0$, a constant $C > 0$, and a continuous positive definite function $\alpha_4 \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for all times $t \in \mathbb{R}$ and all $x^1, x^2 \in \mathbb{R}^n$*

$$\begin{aligned} &(x^1 - x^2)^\top P \left(f(t,x^1) - f(t,x^2)\right) \\ &\leqslant \begin{cases} -\alpha_4(\|x^1 - x^2\|) & \text{if } \max\{\|x^1\|, \|x^2\|\} \geqslant C, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{14}$$

Then (1) *is GIS.*

Examples of systems to which Theorem 10 is applicable include all so-called quadratically convergent systems, see [14], i.e., globally convergent systems where the convergence property is characterized by a quadratic Lyapunov-type function. This also includes systems satisfying the convergence conditions in [23, 7].

On the other hand, it is interesting to ask when an IS system is also convergent. This will be answered in the next section.

3.3. From incremental stability to convergence

Let us recall one of Demidovich's results [7], which can be found as Theorem 1 in [13]. This result provides a sufficiency condition for system (1), with f continuously differentiable in x, being GIS, namely that there exists a positive definite matrix $P = P^\top$ so that

$$J(x,t) = \frac{1}{2}\left[P\frac{\partial f}{\partial x}(t,x) + \Big(\frac{\partial f}{\partial x}(t,x)\Big)^\top P\right] \tag{15}$$

is negative definite uniformly in $(t,x) \in \mathbb{R}^{1+n}$. It also provides a sufficiency condition,

$$\|f(t,0)\| \leqslant c < \infty, \tag{16}$$

following ideas by Yakubovich and Demidovich, which together with (15) guarantee the positive invariance and global asymptotic stability of a compact set $\Omega := \{x \in \mathbb{R}^n : x^\top P x \leqslant C\}$ with a constant C depending on P and c.

Interestingly, condition (15) actually implies that all solutions of (1) are globally uniformly exponentially stable, cf. [13], i.e., it implies even more than GIS. So in light of Example 3 this condition appears to be stronger than required. In effect, this condition imposes the existence of a *quadratic* Lyapunov function $V(x_1 - x_2) = (x_1 - x_2)^\top P(x_1 - x_2)$ on the differences between trajectories. A more general version using Lyapunov functions $V(x_1, x_2)$ of two arguments can be found in [15, Theorem 2.40, p.28]. This general type of Lyapunov function would usually not imply exponential incremental stability, but it still implies GIS.

Below, we present a result that IS on compact sets implies uniform convergence on compact sets, where the implication does not hinge on the existence of certain (incremental) Lyapunov functions.

Theorem 11. *Suppose system* (1) *is incrementally stable in a* compact *set $\mathcal{X}$. Then it is also uniformly convergent in $\mathcal{X}$.*

Finally, we present a result providing conditions under which *global* incremental stability implies *global* uniform convergence. Results tailored specifically to dissipative, periodic systems have been presented in [16]. The result below is formulated for the more general class of time-varying systems of the form (1).

Theorem 12. *Suppose that system* (1) *is GIS, where $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is locally Lipschitz in $x \in \mathbb{R}^n$. Then, the following statements hold:*

1. *There exists a sufficiently small $c \geqslant 0$ such that if $\|f(t,0)\| \leqslant c$, for all $t \in \mathbb{R}$, then system* (1) *is globally uniformly convergent;*

2. *If there exists a compact set $\Omega \subset \mathbb{R}^n$ that is positively invariant with respect to* (1)*, then system* (1) *is globally uniformly convergent.*

The magnitude of $c \geqslant 0$ is a measure of the magnitude of the vector field $f(t,0)$. As we are employing a converse Lyapunov result in the proof of Theorem 12, we cannot provide a more explicit formula that c needs to satisfy. Please note that for $c = 0$ the result is obvious, since then the origin is a globally asymptotically stable equilibrium. However, when a GIS-Lyapunov function is known, the assumption on the vector field can be made more explicit, as in the following corollary.

Corollary 13. *Assume that there exist functions W, α_3, and γ as in Corollary 6. Let $c \geqslant 0$ be such that for all small $h > 0$,*

$$\|f(t,0)\|h \leqslant ch < \gamma^{-1}(h\alpha_3(r)) \tag{17}$$

for some $r > 0$. Then system (1) *is globally uniformly convergent.*

Remark 14. *The existence of a positively invariant compact set in the second statement of Theorem 12 can be inferred from explicit conditions on the vector field f and the* boundary *of a compact candidate set $K \subset \mathbb{R}^n$.*

One such condition, cf. [4, Theorem 5] or [3, Theorem 11.6.2], is that there exists an integrable function $k \in L_1(\mathbb{R},\mathbb{R})$ such that $f\colon \mathbb{R}\times\mathcal{X} \to \mathbb{R}^n$ is Lipschitz with respect to x in the sense that

$$\|f(t,x) - f(t,y)\| \leqslant k(t)\|x-y\|. \tag{18}$$

Then K is positively invariant under (1) *if for all $t \in \mathbb{R}$ and $x \in \partial K$ (the boundary of K),*

$$f(t,x) \in \left\{ v \in \mathbb{R}^n\colon \liminf_{h\searrow 0} \frac{d_K(x+hv)}{h} = 0 \right\} \tag{19}$$

where

$$d_K(y) := \inf_{x\in K} \|y-x\|$$

is the distance from y to K.

Remark 15. *We note that, for smooth systems, the Lyapunov-based sufficient conditions for uniform convergence in [7, 23, 13] are special cases of Theorem 12 in the sense that quadratic Lyapunov functions are employed to characterise incremental stability properties (and the existence of a compact positively invariant set). Hence, the classes of systems treated in these references can be considered examples satisfying the conditions of Theorem 12.*

It should also be noted that in the result of Demidovich [7, 13] also the condition $\|f(t,0)\| \leqslant c$, $\forall t$, with $c > 0$, is employed as in claim 1) in Theorem 12. However, by the grace of the fact that quadratic Lyapunov functions are used in [7, 13] to characterise incremental stability properties, the satisfaction of $\|f(t,0)\| \leqslant c$, $\forall t$, for any $c > 0$ is sufficient to prove global uniform convergence in [7, 13].

4. Conclusions

The global uniform convergence property and global incremental asymptotic stability are very related and yet different properties. This paper in particular contributes examples of systems that are globally uniformly convergent but not globally incrementally stable (and vice versa). These examples further illuminate the essential differences between these stability notions. Moreover, we present results that state sufficient conditions under which the one property implies the other.

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A. Appendix – Proofs and auxiliary results

A.1. Proof of the converse Lyapunov results in Section 3.1

Proof of Theorem 5. The proof is similar to the proof given by Angeli [2], but there are some significant and non-obvious differences that we will elaborate on. The main difference and technical difficulty lies in the fact that while the systems (7) considered in [2] can depend on a time-varying perturbation, they may not depend on time explicitly. In contrast, our characterization of incremental stability is for systems depending explicitly on time. The main differences are thus related to the uniformity of the decay of the Lyapunov function. This boils down to a different definition for $U(t^0, z^0)$ in step 3 of the proof, as compared to Angeli's proof. Another difference is the use of Sontag's Lemma on $\mathcal{KL}$-functions in step 7, where another argument was used in the original proof. Finally, we use a scaling argument similar to the one used in [20] in order to obtain a decay rate of class $\mathcal{K}_\infty$ in step 8.

The 'if'-part of the proof follows standard arguments (see, e.g., [9, Theorem 3.2.7]) and is thus omitted. In the following we treat the 'only if'-part.

Let us adopt the following notation for this proof. We consider

$$\dot{x} = f(t, x) \tag{20}$$

and

$$\dot{z} = \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f(t, x_1) \\ f(t, x_2) \end{pmatrix} \tag{21}$$

as in [2]. We have that the diagonal $\Delta := \{(x^\top, x^\top)^\top : x \in \mathbb{R}^n\} \subset \mathbb{R}^{2n}$ is GAS w.r.t. system (21) if and only if system (20) is GIS, as is shown in Lemma 2.3 in [2][2]. The distance of a point $z = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to the diagonal Δ is given by

$$\|z\|_\Delta := \inf_{w \in \Delta} \|w - z\|$$

and it is shown in [2] that this equals

$$\|z\|_\Delta = \frac{1}{\sqrt{2}}\|x_1 - x_2\|.$$

Now to the details of the proof:

[2] Note that [2, Lemma 2.3] holds also true for (explicitly) time-dependent nonlinear systems (21), although in [2] "disturbance-dependent" systems are considered.

1. First we define
$$g(t^0, z^0) := \sup_{t \geqslant t^0} \|z(t, t^0, z^0)\|_\Delta \tag{22}$$
which satisfies for the $\mathcal{K}_\infty$ functions $\widetilde{\alpha}_1 = \mathrm{id}$ and $\widetilde{\alpha}_2 = \beta(\cdot, 0)$, where β comes from the definition of GIS, the estimate
$$\widetilde{\alpha}_1(\|z\|_\Delta) \leqslant g(t, z) \leqslant \widetilde{\alpha}_2(\|z\|_\Delta) \tag{23}$$
for all $z \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}$. Observe that the supremum in (22) is in fact a maximum, since $\|z(\cdot, t^0, z^0)\|_\Delta$ is continuous and tends to zero as time tends to infinity. The function g also satisfies the continuity property
$$\begin{aligned} |g(t, z^1) - g(t, z^2)| &\leqslant \sqrt{2}\beta(2\|z^1 - z^2\|_\Delta, 0) \\ &=: \widetilde{\gamma}(\|z^1 - z^2\|_\Delta), \end{aligned} \tag{24}$$
for all $z^1, z^2 \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}$. This can be proved as per Fact 2.5 in [2].
2. Along solutions the function g is obviously non-increasing: For $s > 0$ we have
$$g(t^0, z^0) \geqslant g(t^0 + s, z(t^0 + s, t^0, z^0)).$$
3. Now define
$$U(t^0, z^0) := \sup_{s \geqslant 0} g(t^0 + s, z(t^0 + s, t^0, z^0))k(s),$$
where k is any continuously differentiable, positive, increasing function for which there exist $1 \leqslant c_1 < c_2$ such that $k(t) \in [c_1, c_2]$ for all $t \in \mathbb{R}_+$, and the derivative of k is bounded from below by some positive and decreasing function d, i.e. $\dot{k}(t) \geqslant d(t)$ for all $t \in (0, \infty)$. Necessarily $d(t) \to 0$ as $t \to \infty$, since otherwise (and because $d(t) \geqslant 0$) k would grow without bound.
4. In view of $c_2 \geqslant k(t) \geqslant c_1 \geqslant 1$ for all $t \in \mathbb{R}_+$ and (23) it follows that
$$U(t^0, z^0) \geqslant g(t^0, z^0) \geqslant \|z^0\|_\Delta \tag{25}$$
and
$$U(t^0, z^0) \leqslant c_2 \widetilde{\alpha}_2(\|z^0\|_\Delta). \tag{26}$$
Using the relation $\|z\|_\Delta = \frac{1}{\sqrt{2}}\|x_1 - x_2\|$, the inequalities (25) and (26) establish
$$\begin{aligned} &\overline{\alpha}_1(\|x_1 - x_2\|) := \frac{1}{\sqrt{2}}\|x_1 - x_2\| \leqslant U(t^0, x_1, x_2) \text{ and} \\ &U(t^0, x_1, x_2) \leqslant c_2 \widetilde{\alpha}_2\Big(\frac{\|x_1 - x_2\|}{\sqrt{2}}\Big) =: \overline{\alpha}_2(\|x_1 - x_2\|). \end{aligned} \tag{27}$$
5. From the definition of U it follows that for all $t^0 \in \mathbb{R}$ and any $z^1, z^2 \in \mathbb{R}^{2n}$ and for all $\epsilon > 0$ there exists an $s_\epsilon = s_{\epsilon, t^0, z^1} \geqslant 0$ such that
$$U(t^0, z^1) \leqslant \epsilon + g(t^0 + s_\epsilon, z(t^0 + s_\epsilon, t^0, z^1))k(s_\epsilon).$$

This inequality yields, in view of $k(t) \leqslant c_2$ for all $t \in \mathbb{R}_+$ and (24), in a few steps (refer to Angeli's proof in [2]) that

$$U(t^0, z^1) - U(t^0, z^2) \leqslant \epsilon + \widetilde{\gamma}(\beta(\|z^1 - z^2\|, 0))c_2.$$

With ϵ arbitrary and using a symmetry argument we arrive at $|U(t^0, z^1) - U(t^0, z^2)| \leqslant \gamma(\|z^1 - z^2\|)$, where $\gamma(r) = \widetilde{\gamma}\big(\beta(r, 0)\big)c_2$.

6. By definition, U is non-increasing along solutions. We will now show that U strictly decreases along solutions of (21).
   By the definition of U, for all $r > 0$ and $z^0 \in \mathbb{R}^{2n}$ with $\|z^0\|_\Delta = r$, for all $t^0 \in \mathbb{R}$, all $h > 0$, and all $\epsilon > 0$, there exists an $s = s_{\epsilon,h,t^0,z^0} \geqslant 0$ such that we can show that

$$\begin{aligned} &U(t^0 + h, z(t^0 + h, t^0, z^0)) \\ &\quad \leqslant U(t^0, z^0)\left[1 - \frac{k(h+s) - k(s)}{c_2}\right] + \epsilon. \end{aligned} \tag{28}$$

7. Now we would like to let $h \searrow 0$ and $\epsilon \to 0$ in (28) to obtain an estimate on the decay of U along solutions of (21). For this we have to ensure that s in (28) does not grow without bound when ϵ and h tend to zero.
   Claim: For all $r > 0$ there exists a $T = T(r) > 0$ such that s in (28) satisfies $s \leqslant T$, independent of the choice of $h > 0$ and $\epsilon > 0$.
   Proof: We start by recalling a known fact. From Sontag's Lemma on $\mathcal{KL}$-functions [19] it is known that for any $\beta \in \mathcal{KL}$ there exist functions $\kappa_1, \kappa_2 \in \mathcal{K}_\infty$ such that for all $r, t \in \mathbb{R}_+$,

$$\beta(r, t) \leqslant \kappa_1\big(\kappa_2(r)e^{-t}\big). \tag{29}$$

A simple consequence of (29) is that for any $\delta > 0$ we have

$$\beta(r, t) < \delta \text{ whenever } t > \ln \frac{\kappa_2(r)}{\kappa_1^{-1}(\delta)}. \tag{30}$$

Now we prove the claim. We know from estimates (25) and (26) that

$$0 < r = \|z^0\|_\Delta \leqslant U(t^0, z^0) \leqslant c_2\widetilde{\alpha_2}(r).$$

Continuity and monotonicity properties of U along trajectories of (21) with $\|z^0\|_\Delta = r$ yield that for some $\nu > 0$, $\mu > 0$,

$$\begin{aligned} \nu + \epsilon &< U(t^0, z^0) - \mu \\ &< U(t^0 + h, z(t^0 + h, t^0, z^0)) \\ &\leqslant U(t^0, z^0) \end{aligned} \tag{31}$$

for all $0 < h < \overline{h} = \overline{h}(\epsilon)$ if $\epsilon > 0$ is sufficiently small, which we will henceforth assume.

Let $\delta = \nu/c_2$ and let us assume that no finite $T > 0$ as in the claim exists. Then for every integer $N > 0$ there must exist an $s > N$ such that (28) holds for this s, i.e., we can show that

$$\begin{aligned} U(t^0 + h, z(t^0 + h, t^0, z^0)) &\leqslant \beta(\|z^0\|_\Delta, h + s)c_2 + \epsilon \\ &< \nu + \epsilon \text{ whenever } s > \ln \frac{\kappa_2(r)}{\kappa_1^{-1}(\nu/c_2)} \text{ due to (30).} \end{aligned}$$

Considering (31) we arrive at the contradiction

$$\nu + \epsilon < U(t^0 + h, z(t^0 + h, t^0, z^0)) < \nu + \epsilon$$

thus proving the claim. □

Hence we have shown that we can pass to an appropriate limit in (28) as $h \searrow 0$ and $\epsilon \to 0$, since $s = s_{\epsilon,h,t^0,z^0}$ in (28) remains bounded.

8. Following essentially the same arguments as in [2] we obtain for some positive definite function $\widetilde{\alpha}_3$,

$$\begin{aligned} \dot{U}(t^0, z^0) &:= \limsup_{h \searrow 0} \\ &\frac{U(t^0 + h, z(t^0 + h, t^0, z^0)) - U(t^0, z^0)}{h} \\ &\leqslant -\widetilde{\alpha}_3(\|z^0\|_\Delta). \end{aligned}$$

At this stage it is left to show that we can modify U such that the function $\widetilde{\alpha}_3$ can be taken to be of class $\mathcal{K}_\infty$. The argument we are going to make follows the idea in [20].

To this end let $\mu, \rho \in \mathcal{K}_\infty$ such that $\rho' = \mu$ and that $s \mapsto (\mu \circ \alpha_1^{-1})(s)\widetilde{\alpha}_3(s)$ is bounded from below by some class $\mathcal{K}_\infty$ function α_3. This is always possible.

Define $W := \rho(U)$ and verify using (27) that it satisfies bounds (8) with $\alpha_i = \rho \circ \overline{\alpha}_i$, $i = 1, 2$. Compute

$$\begin{aligned} \dot{W}(t^0,z^0) &:= \limsup_{h \searrow 0} \\ &\frac{W(t^0 + h, z(t^0 + h, t^0, z^0)) - W(t^0, z^0)}{h} \\ &= \limsup_{h \searrow 0} \rho'(U(\tau_{t^0,h}, z(\tau_{t^0,h}, t^0, z^0)))\cdot \\ &\frac{U(t^0 + h, z(t^0 + h, t^0, z^0)) - U(t^0, z^0)}{h} \\ &\leqslant -\rho'\big(\overline{\alpha}_1^{-1}(\|z^0\|_\Delta)\big) \cdot \widetilde{\alpha}_3(\|z^0\|_\Delta) \\ &\leqslant -\alpha_3(\|z^0\|_\Delta), \end{aligned} \tag{32}$$

with $\widetilde{\alpha}_3 \in \mathcal{K}_\infty$ and where in equation (32) we have used the mean value theorem to obtain a sequence $\tau_{t^0,h} \xrightarrow{h \to 0} t^0$ of points in $(t^0, t^0 + h)$, followed by continuity of ρ' and U with respect to time.

9. Now, following again the same arguments as in [2] we obtain for $t \geqslant t^0$, $W(t, z(t,t^0,z^0)) - W(t^0,z^0) \leqslant -\int_{t^0}^{t} \alpha_3(\|z(s,t^0,z^0)\|_\Delta)\mathrm{d}s$, which proves the inequality (9) in the theorem. This completes the proof of the theorem. □

Proof of Corollary 6. Just take instead of W the function U defined in the preceding proof at the end of step 5, it satisfies all the requirements by construction. Without loss of generality, the function γ can be taken to be class $\mathcal{K}_\infty$. □

A.2. Proofs of the results in Section 3.2 (From convergence to incremental stability)

Proof of Theorem 8. For future reference we denote $d_{\mathcal{X}} := \max_{x,y\in\mathcal{X}} \|x-y\|$, the diameter of $\mathcal{X}$. Note that without loss of generality we can assume that the closure of the trajectory $\overline{x}$ (which is a compact set) is contained in $\mathcal{X}$, i.e., $\overline{\bigcup_{t\in\mathbb{R}}\{\overline{x}(t)\}} \subset \mathcal{X}$.

We are going to show that differences of solutions satisfy the uniform attraction and stability properties for restricted initial conditions.

Uniform attraction: For any $\epsilon > 0$ there exists a $T > 0$ such that for any $\xi \in \mathcal{X}$, $\|x(t,t^0,\xi) - \overline{x}(t)\| \leqslant \beta(d_{\mathcal{X}}, t-t^0) \leqslant \epsilon/2$ if $t - t^0 \geqslant T$. By the triangle inequality it follows that for any $\xi, \eta \in \mathcal{X}$, $\|x(t,t^0,\xi) - x(t,t^0,\eta)\| \leqslant \epsilon$ if $t - t^0 \geqslant T$. This shows that all solutions starting in $\mathcal{X}$ are mutually uniformly attractive.

Uniform stability: The following argument follows ideas in the proof of [18, Theorem 55]. Let $\xi^1, \xi^2 \in \mathcal{X}$ and $t^0 \in \mathbb{R}$ be arbitrary. In view of item 3 of Definition 1 we have that $\|x(t,t^0,\xi^1) - x(t,t^0,\xi^2)\| \leqslant 2\beta(d_{\mathcal{X}}, t-t^0)$ for all $t > t^0$, i.e., there exists a $\mathcal{KL}$ function $\widehat{\beta}$ such that

$$\|x(t,t^0,\xi^1) - x(t,t^0,\xi^2)\| \leqslant \widehat{\beta}(d_{\mathcal{X}}, t-t^0) \text{ for all } t > t^0.$$

Thus there exists a compact set $\mathcal{Y} \supset \mathcal{X}$ which contains all solutions with initial values in $\mathcal{X}$ (in fact, $\mathcal{X}$ is positively invariant, so $\mathcal{Y} = \mathcal{X}$). Write $x^1(t) := x(t,t^0,\xi^1)$ and $x^2(t) := x(t,t^0,\xi^2)$. Regarding

$$x^1(t) - x^2(t) = \xi^1 - \xi^2 + \int_{t^0}^{t} [f(s,x^1(s)) - f(s,x^2(s))]\mathrm{d}s$$

for all $t \geqslant t^0$, we have due to the local Lipschitz condition on f and the compactness of $\mathcal{X}$ that there exists a locally integrable function $\alpha\colon \mathbb{R} \to \mathbb{R}_{\geqslant 0}$, cf. [18, Appendix C], such that for all $t \geqslant t^0$,

$$\|x^1(t) - x^2(t)\| \leqslant \|\xi^1 - \xi^2\| + \int_{t^0}^{t} \alpha(s)\|x^1(s) - x^2(s)\|\mathrm{d}s$$

Thus, with Gronwall's inequality we arrive at

$$\|x^1(t) - x^2(t)\| \leqslant \|\xi^1 - \xi^2\| e^{\left(\int_{t^0}^{t} \alpha(s)\mathrm{d}s\right)}$$

for all $t \geqslant t^0$. As $\|x^1(t) - x^2(t)\| \leqslant \widehat{\beta}(d_{\mathcal{X}}, t-t^0)$ for all $t \geqslant t^0$, we arrive at

$$\|x^1(t) - x^2(t)\| \leqslant \min\left\{\|\xi^1 - \xi^2\| e^{\left(\int_{t^0}^{t} \alpha(s)\mathrm{d}s\right)}, \widehat{\beta}(d_{\mathcal{X}}, t-t^0)\right\}.$$

From there we can obtain a $\mathcal{KL}$ function $\widetilde{\beta}$ such that

$$\|x(t,t^0,\xi^1)-x(t,t^0,\xi^2)\| \leqslant \widetilde{\beta}(\|\xi^1-\xi^2\|, t-t^0)$$

for all $\xi^1,\xi^2 \in \mathcal{X}$, $t^0 \in \mathbb{R}$ and $t \geqslant t^0$. □

Proof of Theorem 10. By Theorem 7, which is about the characterization of the uniform convergence property, there exists a Lyapunov function V satisfying (11) and (12). The solution $\overline{x}$ is bounded on $\mathbb{R}$, i.e. there exists a $C_2 \geqslant 0$ such that $\|\overline{x}(t)\| \leqslant C_2$ for all $t \in \mathbb{R}$. Without loss of generality we can assume that $C - C_2 > 0$, if necessary by enlarging C for which (14) is satisfied. There also exist positive constants c_P, C_P such that for all $x^1, x^2 \in \mathbb{R}^n$, $c_P\|x^1-x^2\|^2 \leqslant (x^1-x^2)^\top P(x^1-x^2) \leqslant C_P\|x^1-x^2\|^2$.

Denote $K := \{(x^1,x^2) \in \mathbb{R}^n \times \mathbb{R}^n \colon \max\{\|x^1\|,\|x^2\|\} \leqslant C\}$. On the compact set K we have $V(t,x^1)+V(t,x^2) \leqslant \alpha_2(\|x^1-\overline{x}(t)\|)+\alpha_2(\|x^2-\overline{x}(t)\|) \leqslant 2\alpha_2(C+C_2)$, where V is given by Theorem 7.

Let us define $W(t,x^1,x^2) := \frac{1}{2}b\big(V(t,x^1)+V(t,x^2)\big)\ (x^1-x^2)^\top P(x^1-x^2)$ where $b(s) = s/(1+s)$ is a bounded class $\mathcal{K}$ function. We have

$$\begin{aligned} W(t,x^1,x^2) &\leqslant \frac{1}{2}C_P\|x^1-x^2\|^2 \\ &=: \widetilde{\alpha}_2(\|x^1-x^2\|) \end{aligned}$$

since $b(s) \leqslant 1$ for all $s \geqslant 0$. We also have that

$$\begin{aligned} W(t,x^1,x^2) &\geqslant \frac{1}{2}b\big(\alpha_1(\|x^1-\overline{x}\|) \\ &\qquad + \alpha_1(\|x^2-\overline{x}\|)\big)c_P\|x^1-x^2\|^2 \\ &\geqslant \frac{1}{2}b\bigg(\alpha_1\bigg(\frac{1}{2}\|x^1-\overline{x}\| \\ &\qquad + \frac{1}{2}\|x^2-\overline{x}\|\bigg)\bigg)c_P\|x^1-x^2\|^2 \\ &\geqslant \frac{1}{2}b\left(\alpha_1\left(\frac{\|x^1-x^2\|}{2}\right)\right)c_P\|x^1-x^2\|^2 \\ &=: \widetilde{\alpha}_1(\|x^1-x^2\|). \end{aligned}$$

So W is positive definite and radially unbounded in the distance $\|x^1-x^2\|$.

Denoting $\dot{V}(x^i) := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x^i) \leqslant -\alpha_3(\|x^i-\overline{x}(t)\|)$ as per (12) and $\frac{d}{ds}b(s)$ by $b'(s)$, we compute the time-derivative of W as

$$\begin{aligned} \dot{W} &:= \frac{d}{dt}W(t,x^1(t),x^2(t)) \\ &= b'\big(V(t,x^1)+V(t,x^2)\big)[\dot{V}(x^1)+\dot{V}(x^2)] \\ &\qquad \cdot \frac{1}{2}(x^1-x^2)^\top P(x^1-x^2) \\ &\quad + b\big(V(t,x^1)+V(t,x^2)\big) \end{aligned}$$

$$\cdot (x^1 - x^2)^\top P \left(f(t, x^1) - f(t, x^2) \right). \tag{33}$$

On the set K, the first term in the right-hand side of (33) is bounded from above by

$$-\frac{1}{2}\alpha_3 \left(\frac{\|x^1 - x^2\|}{2} \right) \frac{c_P \|x^1 - x^2\|^2}{\big(1 + 2\alpha_2(C + C_2)\big)^2}$$

while the second term in the right-hand side of (33) is nonpositive due to (14). Outside of K the first term could be arbitrarily small in magnitude as $b'(s) \to 0$ for $s \to \infty$, while this term is still negative. Hence, outside of K, (33) is bounded from above by

$$\begin{aligned} &b\big(2\alpha_1(C - C_2)\big)(x^1 - x^2)^\top P \left(f(t, x^1) - f(t, x^2) \right) \\ &\leqslant -\alpha_4(\|x^1 - x^2\|) b\big(2\alpha_1(C - C_2)\big), \end{aligned}$$

again due to (14). It follows that $\dot{W}$ is bounded from above by a function which is negative definite with respect to the set where $x^1 = x^2$. A standard scaling argument (see [20]) with $U = \rho(W)$ for a suitable function $\rho \in \mathcal{K}_\infty$ turns this into a smooth Lyapunov function satisfying $\dot{U} \leqslant -\alpha_5(U)$ with $\alpha_5 \in \mathcal{K}_\infty$. This function U in particular satisfies (8) and (9). Hence, by virtue of Theorem 5 we conclude that system (1) is indeed GIS. □

A.3. Proofs of the results in Section 3.3 (From incremental stability to convergence)

We start with an auxiliary result.

Proposition 16. *Let $A \subset \mathbb{R}^n$ be a compact and positively invariant set for system* (1)*. Then there exists a solution $\overline{x}(t)$ in A which is defined for all times.*

The proof is a simplified version of [23, Lemma 2] and omitted for the sake of brevity.

Proof of Theorem 11. By Proposition 16 there exists a bounded solution $\overline{x}(t)$ in $\mathcal{X}$ which is defined for all times. As all solutions are uniformly attractive, so is $\overline{x}(t)$.

The uniqueness proof follows the same reasoning as the proof of Property 2.4 in [15]. □

Proof of Theorem 12. Let us first show that the condition in claim 1. in the theorem together with the fact that the system is GIS implies the existence of a compact positively invariant set $\Omega \subset \mathbb{R}^n$.

According to Corollary 6, the fact that the system is GIS implies that there exists a continuous function $W(t, x_1, x_2)$ satisfying (8) and (9) with $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and α_3 positive definite and there exists a function $\gamma \in \mathcal{K}_\infty$ such that (10) holds.

With $V(t, x) := W(t, x, 0)$ and for $h > 0$ sufficiently small, we compute

$$V(t^0 + h, x(t^0 + h, t^0, \xi)) - V(t^0, \xi)$$

$$
\begin{aligned}
&= W(t^0+h, x(t^0+h,t^0,\xi),0) - W(t^0,\xi,0)\\
&= W(t^0+h, x(t^0+h,t^0,\xi),0) - W(t^0,\xi,0)\\
&\quad + W(t^0+h, x(t^0+h,t^0,\xi), x(t^0+h,t^0,0))\\
&\quad - W(t^0+h, x(t^0+h,t^0,\xi), x(t^0+h,t^0,0))\\
&\overset{(9)}{\leqslant} -\int_{t^0}^{t^0+h} \alpha_3\big(\|x(\tau,t^0,\xi)-x(\tau,t^0,0)\|\big)d\tau \qquad (34)\\
&\quad \left.\begin{aligned}&+ W(t^0+h, x(t^0+h,t^0,\xi),0)\\ &- W(t^0+h, x(t^0+h,t^0,\xi), x(t^0+h,t^0,0)).\end{aligned}\right\} \qquad (35)
\end{aligned}
$$

The last inequality proves the existence of a compact positively invariant set if the term (34) dominates the term (35), so that the entire expression becomes negative for large enough ξ.

For (35) we compute, using the Landau symbol $\mathcal{O}$,

$$
\begin{aligned}
&W(t^0+h, x(t^0+h,t^0,\xi),0)\\
&\quad - W(t^0+h, x(t^0+h,t^0,\xi), x(t^0+h,t^0,0))\\
&\quad \overset{(10)}{\leqslant} \gamma\left(\left\|\begin{pmatrix} x(t^0+h,t^0,\xi)\\ 0\end{pmatrix} - \begin{pmatrix} x(t^0+h,t^0,\xi)\\ x(t^0+h,t^0,0)\end{pmatrix}\right\|\right)\\
&\quad = \gamma(\|x(t^0+h,t^0,0)\|) \leqslant \gamma(\|f(t^0,0)\|h + \mathcal{O}(h^2))\\
&\quad \leqslant \gamma(ch+\mathcal{O}(h^2)) =: C.
\end{aligned}
$$

For large $\xi \in \mathbb{R}^n$ the integral in (34) dominates C, if c is chosen sufficiently small. In this case we have $\dot{W} < 0$ outside a compact set, rendering the said compact set positively invariant.

Now, we have that under the conditions of claims 1. and 2. in the theorem, there exists a compact positively invariant set for system (1). By Proposition 16, the existence of a compact positively invariant set implies the existence of a solution $\bar{x}(t)$ which is defined and bounded for all $t \in \mathbb{R}$.

This solution $\overline{x}(t)$ is uniformly globally asymptotically stable, since all solutions are uniformly globally asymptotically stable (since the system is GIS by assumption). From here it follows that $\overline{x}(t)$ must also be unique, see [15, p.15, Property 2.15]. This completes the proof. □

Proof of Corollary 13. We use the same notation as in the previous proof. If $c \geqslant 0$ is chosen such that (17) holds then the increments of V are non-positive outside a compact set in the preceding proof. □