Incremental Stability Properties for Discrete-Time Systems

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***Abstract*— Incremental stability describes the asymptotic behavior between any two trajectories of a dynamical system. Such properties are of interest, for example, in the study of observers or synchronization of chaotic systems. In this paper, we develop the notions of incremental stability and incremental input-to-state stability (ISS) for discrete-time systems. We derive Lyapunov function characterizations for these properties as well as a useful summation-to-summation formulation of the incremental stability property.**

I. Introduction

Incremental stability [1] (referred to as extreme stability in [17]) extends the classical notion of asymptotic stability of an equilibrium of a nonlinear system to consider the asymptotic behavior of any solution with respect to any other solution. Specifically, any two solutions must eventually asymptotically converge to each other regardless of their initial conditions. Incremental stability is not the only notion to describe the asymptotic behavior of a solution with respect to other solutions. Contraction analysis and convergent dynamics provide two additional methods to characterize such a property, see [8] and [10], respectively, (see also [11] for a comparison between incremental stability and convergent systems), and references therein.

Introduced by Sontag [12], Input-to-State Stability (ISS) has proven to be one of the most useful robust-stability analysis tools for nonlinear systems. Subsequent developments enabled the study of different relationships between input, output, and state, including incremental ISS. See [14] for a comprehensive survey of ISS notions. Informally, if a system satisfies the incremental ISS property and the differences between two input signals are small and bounded, then the distance between any two trajectories must eventually be small and independent of initial conditions.

Motivated by the corresponding incremental stability and incremental ISS notions proposed in [1] for continuous-time nonlinear systems, in this paper we study such properties for discrete-time systems. Such incremental properties for both systems with and without inputs are useful in studying problems such as observer analysis, controller design, and chaos synchronization (c.f., [1], [18], and [19]).

The contribution of this paper is two-fold. First, incremental stability of discrete-time systems without input is considered where we demonstrate a Lyapunov function characterization analogous to [1, Theorem 1]. In addition, we show that incremental stability is equivalent to a summation-to-summation estimate which provides further insights to the incremental stability property. Second, we consider discrete-time nonlinear systems with inputs and the incremental ISS property. We present a forward Lyapunov (dissipation) function result for incremental ISS. Then, aiming for a converse result, we compare two different forms of an incremental ISS Lyapunov function, namely the dissipation-form and the implication-form, and provide a necessary condition where the existence of one of those functions implies the existence of the other one. Following the standard discrete-time ISS result in [2, Theorem 1], with appropriate assumptions, we demonstrate various incremental properties equivalent to incremental ISS including a separation principle, a robust feedback stability notion and, most importantly, a Lyapunov function characterization; all in the incremental sense.

The paper is organized as follows: the necessary technical assumptions and notational conventions are provided in Section II. In Section III, the incremental stability and incremental summation-to-summation notions are defined, then a Lyapunov function characterization for those equivalent notions are presented. In Section IV, following [2, Theorem 1], similar characterizations are presented in the incremental ISS context, where the sufficiency of certain assumptions on the sets of input and state are described (where needed). Conclusions are provided in Section V and several proofs are collected in the appendix.

II. Preliminaries

Let $\mathbb{G} \subseteq \mathbb{R}^n$, $\mathbb{W} \subseteq \mathbb{R}^m$ with $0 \in \mathbb{G}$ and $0 \in \mathbb{W}$. We consider discrete-time nonlinear systems described by the difference equation

$$x(k+1) = f(x(k)), \quad x(k) \in \mathbb{G}, k \in \mathbb{Z}_{\geq 0} \tag{1}$$

where $f : \mathbb{G} \to \mathbb{G}$ is continuous and $f(0) = 0$. We also consider systems with inputs described by

$$x(k+1) = f(x(k), w(k)), \quad x(k) \in \mathbb{G}, k \in \mathbb{Z}_{\geq 0} \tag{2}$$

where $f : \mathbb{G} \times \mathbb{W} \to \mathbb{G}$ is continuous, $f(0,0) = 0$, and $w : \mathbb{Z}_{\geq 0} \to \mathbb{W}$ denotes an input sequence. We denote the set of admissible input sequences by $\mathcal{W}$. We denote $x : \mathbb{Z}_{\geq 0} \times \mathbb{G} \to \mathbb{G}$ as the solution of system (1) from an initial condition $\xi \in \mathbb{G}$; i.e., a function satisfying $x(0, \xi) = \xi$ and the dynamics (1). With a slight abuse of notation, we also denote $x : \mathbb{Z}_{\geq 0} \times \mathbb{G} \times \mathcal{W} \to \mathbb{G}$ as the solution of system (2) from an initial condition $\xi \in \mathbb{G}$, subject to input sequence $w \in \mathcal{W}$; i.e., a function satisfying $x(0, \xi, w) = \xi$ and the dynamics (2). We use standard comparison function classes $\mathcal{K}$, $\mathcal{L}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$. See [4] for details about comparison

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functions[1]. For $y \in \mathbb{R}^n$, we denote the supremum norm by $\|z\|_\infty := \sup\{|z(k)| : k \in \mathbb{Z}_{\geq 0}\}$. For $K \in \mathbb{Z}_{\geq 0}$, we denote $\mathbb{Z}_{\geq K} := \{k \in \mathbb{Z} : k \geq K\}$.

Associated with (1), we also consider the augmented system

$$\begin{aligned} x_1(k+1) &:= f(x_1(k)) \\ x_2(k+1) &:= f(x_2(k)) \end{aligned} \tag{3}$$

as in [1]. Let the diagonal set be $\Delta := \{(x^T, x^T)^T : x \in \mathbb{R}^n\}$. The distance to Δ from a point $z = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$, $x_1, x_2 \in \mathbb{R}^n$, is given by

$$|z|_\Delta := \sup_{w \in \Delta} |w - z| = \frac{1}{\sqrt{2}}|x_1 - x_2|, \tag{4}$$

where the equality is shown in [1].

Intuitively, the augmented system is formed by two copies of the original system. Then, global asymptotic stability of the diagonal set Δ with respect to the augmented system can be demonstrated to be equivalent to incremental stability of the original system via (4). Thus, a Lyapunov characterization for incremental stability of the original system can be derived from a classical Lyapunov characterization of global asymptotic stability for a closed, but unbounded, set.

III. Incremental Stability

We first consider discrete-time systems without input given by (1).

***Definition** 1:* System (1) is globally incrementally asymptotically stable (globally incrementally AS) if there exists $\beta \in \mathcal{KL}$ such that

$$|x(k, \xi_1) - x(k, \xi_2)| \leq \beta(|\xi_1 - \xi_2|, k), \tag{5}$$

holds for all $\xi_1, \xi_2 \in \mathbb{G}$ and $k \in \mathbb{Z}_{\geq 0}$.

***Remark** 2:* Incremental stability, or extreme stability [17], is a special case of stability with respect to two measures. See [5], [9], or [15] for details, including Lyapunov function characterizations. In the two measurement function setting, (5) is called $\mathcal{KL}$-stability with respect to $(|x_1 - x_2|, |x_1 - x_2|)$.

***Definition** 3:* System (1) is globally incrementally α-summable if there exist $\alpha, \eta \in \mathcal{K}_\infty$ such that

$$\sum_{j=0}^{k} \alpha\Big(|x(j, \xi_1) - x(j, \xi_2)|\Big) \leq \eta(|\xi_1 - \xi_2|), \tag{6}$$

holds for all $\xi_1, \xi_2 \in \mathbb{G}$ and $k \in \mathbb{Z}_{\geq 0}$.

As a main tool in classical stability analysis, Lyapunov functions provide a useful method for demonstrating stability without the need to directly solve the difference equation.

***Definition** 4:* An incremental Lyapunov function for system (1) is a smooth function $V : \mathbb{G} \times \mathbb{G} \to \mathbb{R}_{\geq 0}$, and functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|\xi_1 - \xi_2|) \leq V(\xi_1, \xi_2) \leq \alpha_2(|\xi_1 - \xi_2|), \tag{7}$$

$$V\Big(f(\xi_1), f(\xi_2)\Big) - V(\xi_1, \xi_2) \leq -\alpha_3(|\xi_1 - \xi_2|) \tag{8}$$

hold for all $\xi_1, \xi_2 \in \mathbb{G}$ and $k \in \mathbb{Z}_{\geq 0}$.

The following theorem presents an equivalent Lyapunov function characterization for both incremental stability and incremental α-summability.

***Theorem** 5:* The following statements are equivalent:

1) System (1) is globally incrementally AS.
2) System (1) is globally incrementally α-summable.
3) System (1) admits an incremental Lyapunov function.

The proof of Theorem 5 is in Appendix V-A.

IV. Incremental Input-to-State Stability

In this section, we consider discrete-time systems with input described by (2). We first introduce the incremental ISS concept and present a forward incremental Lyapunov function result. Then, following [1, Theorem 2] and [2, Theorem 1], we demonstrate the equivalence between various properties and incremental ISS.

***Definition** 6:* System (2) is globally incrementally Input-to-State Stable (globally incrementally ISS) if there exist functions $\beta \in \mathcal{KL}$, and $\sigma \in \mathcal{K}_\infty$ such that

$$\begin{aligned} &|x(k, \xi_1, w_1) - x(k, \xi_2, w_2)| \\ &\qquad \leq \beta(|\xi_1 - \xi_2|, k) + \sigma(\|w_1 - w_2\|_\infty) \end{aligned} \tag{9}$$

holds for all $\xi_1, \xi_2 \in \mathbb{G}$, $w_1, w_2 \in \mathcal{W}$, and all $k \in \mathbb{Z}_{\geq 0}$.

Similar to the standard ISS and other classical stability notions, we can define a (dissipative) incremental ISS-Lyapunov function whose existence can be used to verify that a particular nonlinear system is incrementally ISS.

***Definition** 7:* A dissipation-form incremental ISS-Lyapunov function for system (2) is a continuous function $V : \mathbb{G} \times \mathbb{G} \to \mathbb{R}_{\geq 0}$ for which there exist functions $\alpha_1, \alpha_2, \alpha_3, \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that

$$\alpha_1(|\xi_1 - \xi_2|) \leq V(\xi_1, \xi_2) \leq \alpha_2(|\xi_1 - \xi_2|), \tag{10}$$

$$\begin{aligned} &V(f(\xi_1, w_1), f(\xi_2, w_2)) - V(\xi_1, \xi_2) \\ &\qquad \leq -\alpha_3(|\xi_1 - \xi_2|) + \sigma(|w_1 - w_2|) \end{aligned} \tag{11}$$

hold for all $\xi_1, \xi_2 \in \mathbb{G}$ and $w_1, w_2 \in \mathbb{W}$.

Now, we are ready to state the forward incremental ISS-Lyapunov function result.

***Theorem** 8:* If system (2) admits a dissipation-form incremental ISS-Lyapunov function then it is incrementally ISS.

The proof of Theorem 8 is in Appendix V-B.

***Remark** 9:* Note that this forward Lyapunov function result holds without assuming that the input set $\mathbb{W} \subseteq \mathbb{R}^m$ is compact. For the converse result shown in the sequel, compactness of $\mathbb{W}$ is required.

A. Dissipation-form vs. Implication-form Incremental ISS-Lyapunov Functions

A standard alternative to the above dissipation-form ISS-Lyapunov function is the following implication-form.

***Definition** 10:* An implication-form incremental ISS-Lyapunov function for system (2) is a continuous function

[1]Recall that $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. If $\alpha \in \mathcal{K}$ is unbounded, it is of class-$\mathcal{K}_\infty$. A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{L}$ if it is continuous, strictly decreasing, and $\lim_{t \to \infty} \sigma(t) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if it is class-$\mathcal{K}$ in its first argument and class-$\mathcal{L}$ in its second argument. By convention, $\beta \in \mathcal{KL}$ satisfies $\beta(0, t) = 0$ for all $t \in \mathbb{R}_{\geq 0}$.

$W : \mathbb{G} \times \mathbb{G} \to \mathbb{R}_{\geq 0}$ for which there exist functions $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3 \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}$ such that

$$\hat{\alpha}_1(|\xi_1 - \xi_2|) \leq W(\xi_1, \xi_2) \leq \hat{\alpha}_2(|\xi_1 - \xi_2|), \tag{12}$$

$$\chi(|\xi_1 - \xi_2|) \geq |w_1 - w_2| \Rightarrow$$
$$W(f(\xi_1, w_1), f(\xi_2, w_2)) - W(\xi_1, \xi_2) \leq -\hat{\alpha}_3(|\xi_1 - \xi_2|) \tag{13}$$

hold for all $\xi_1, \xi_2 \in \mathbb{G}$, $w_1, w_2 \in \mathbb{W}$ and $k \in \mathbb{Z}_{\geq 0}$.

***Lemma** 11:* If system (2) admits a dissipation-form incremental ISS-Lyapunov function (10)–(11), then it admits an implication-form incremental ISS-Lyapunov function (12)–(13).

Proof: Let $V : \mathbb{G} \times \mathbb{G} \to \mathbb{R}_{\geq 0}$ be a dissipation-form incremental ISS-Lyapunov function for (2) with $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ satisfying (10)–(11). Define $\chi \in \mathcal{K}$ by $\chi(s) := \sigma^{-1} \circ \left(\frac{1}{2}\alpha_3\right)(s)$ for all $s \in \mathbb{R}_{\geq 0}$. Then $\chi(|\xi_1 - \xi_2|) \geq |w_1 - w_2|$ and (11) imply

$$\begin{aligned} &V(f(\xi_1, w_1), f(\xi_2, w_2)) - V(\xi_1, \xi_2) \\ &\leq -\alpha_3(|\xi_1 - \xi_2|) + \sigma\big(\sigma^{-1} \circ \left(\tfrac{1}{2}\alpha_3\right)(|\xi_1 - \xi_2|)\big) \\ &\leq -\tfrac{1}{2}\alpha_3(|\xi_1 - \xi_2|). \end{aligned}$$

Hence, V is an implication-form incremental ISS-Lyapunov function for (2). ■

***Remark** 12:* Note that Lemma 11 does not require that $\mathbb{W} \subseteq \mathbb{R}^m$ be compact.

In the standard discrete-time ISS framework, it is known that an implication-form ISS-Lyapunov function exists if and only if a dissipation-form ISS-Lyapunov function exists (see [2, Remark 3.3]). In the incremental context, the converse of Lemma 11 remains an open question. For the purposes of this paper, in order to obtain a partial converse, we follow a result in [16, Proposition 3.2] and require an additional assumption.

***Lemma** 13:* Assume $\mathbb{W} \subseteq \mathbb{R}^m$ and $\mathbb{G} \subseteq \mathbb{R}^n$ are compact. If system (2) admits an implication-form incremental ISS-Lyapunov function (12)–(13), then it admits a dissipation-form incremental ISS-Lyapunov function (10)–(11).

Proof: First notice that if $\chi(|\xi_1 - \xi_2|) \geq |w_1 - w_2|$ then the dissipation-form holds for any choice of $\sigma \in \mathcal{K}$. If $\chi(|\xi_1 - \xi_2|) \leq |w_1 - w_2|$, define a continuous function $\hat{\sigma} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as

$$\begin{aligned} \hat{\sigma}(s) := \max\Big\{ &W(f(\xi_1, w_1), f(\xi_2, w_2)) - W(\xi_1, \xi_2) \\ &+ \hat{\alpha}_3(|\xi_1 - \xi_2|) : |w_1 - w_2| \leq s, \chi(|\xi_1 - \xi_2|) \leq s \Big\}. \end{aligned}$$

Note that $\hat{\sigma}(0) = 0$.

Let $\sigma \in \mathcal{K}_\infty$ satisfy $\sigma(s) \geq \max\{s, \hat{\sigma}(s)\}$ for all $s \in \mathbb{R}_{\geq 0}$. By the definition of σ, it follows that for any $\xi_1, \xi_2 \in \mathbb{G}$,

$$\begin{aligned} \sigma(s) \geq \max_{s = |w_1 - w_2|} \Big\{ &W(f(\xi_1, w_1), f(\xi_2, w_2)) - W(\xi_1, \xi_2) \\ &+ \hat{\alpha}_3(|\xi_1 - \xi_2|) \Big\}. \end{aligned} \tag{14}$$

Note that the maximum exists since W, f, and $\hat{\alpha}_3$ are continuous functions and $\mathbb{W}$ and $\mathbb{G}$ are compact. Therefore, (14) implies

$$\begin{aligned} &\sigma(|w_1 - w_2|) - \hat{\alpha}_3(|\xi_1 - \xi_2|) \\ &\quad \geq W(f(\xi_1, w_1), f(\xi_2, w_2)) - W(\xi_1, \xi_2). \end{aligned} \tag{15}$$

Therefore, W is a dissipation-form ISS-Lyapunov function for (2). ■

***Remark** 14:* In the case of ISS with respect to two measurement functions [16] (see also [7]), examples show that if the level set of the measurement function is noncompact, the existence of an implication-form ISS-Lyapunov function does not imply the existence of a dissipation-form ISS-Lyapunov function, and hence, does not imply ISS with respect to two measurement functions. Generally speaking, incremental ISS is only different from ISS with respect to two measurement functions in consideration of incremental differences of two inputs rather than an individual input. However, it is still unknown if, in the incremental context, the compactness of the set $\mathbb{G}$ in Lemma 13 is required.

As a consequence of Theorem 8, we arrive at the following (stronger) forward incremental ISS-Lyapunov function result.

***Theorem** 15:* Assume $\mathbb{W} \subseteq \mathbb{R}^m$ and $\mathbb{G} \subseteq \mathbb{R}^n$ are compact. If system (2) admits an implication-form incremental ISS-Lyapunov function then it is incrementally ISS.

Proof: The result is a direct consequence of Lemma 13 and Theorem 8. ■

B. A Converse Lyapunov Theorem for Incremental ISS

In this section, we demonstrate a converse Lyapunov theorem for incremental ISS via two additional properties, namely an incremental separation principle and incremental robust feedback stability.

B.1 Separation Principle

In nonlinear stability analysis, it can be useful to separate the effect of the input and the initial condition on the state trajectory in the asymptotic sense.

***Definition** 16:* System (2) has an incremental asymptotic gain if there exists $\gamma \in \mathcal{K}$ such that

$$\begin{aligned} &\limsup_{k \to \infty} |x(k, \xi_1, w_1) - x(k, \xi_2, w_2)| \\ &\qquad \leq \gamma\Big(\limsup_{k \to \infty} |w_1(k) - w_2(k)|\Big) \end{aligned} \tag{16}$$

holds for all $\xi_1, \xi_2 \in \mathbb{G}$, $w_1, w_2 \in \mathcal{W}$, and $k \in \mathbb{Z}_{\geq 0}$.

***Definition** 17:* System (2) is incrementally uniformly bounded input bounded state (incrementally UBIBS) if there exist $\sigma, \gamma \in \mathcal{K}$ such that

$$\begin{aligned} &\sup_{k \in \mathbb{Z}_{\geq 0}} |x(k, \xi_1, w_1) - x(k, \xi_2, w_2)| \\ &\qquad \leq \sigma(|\xi_1 - \xi_2|) + \gamma(\|w_1 - w_2\|_\infty) \end{aligned} \tag{17}$$

holds for all $\xi_1, \xi_2 \in \mathbb{G}$, $w_1, w_2 \in \mathcal{W}$, and $k \in \mathbb{Z}_{\geq 0}$.

Similar to a standard ISS result [2, Lemma 3.8], incremental ISS implies incremental asymptotic gain and incremental UBIBS properties. Note that this implication requires neither a compactness and convexity assumption on the set of inputs nor compactness of the set of states.

***Lemma** 18:* If (2) is incrementally ISS then it has an incremental asymptotic gain and is incrementally UBIBS.

The proof of Lemma 18 is omitted due to space constraints.

B.2 Incremental Robust (Feedback) Stability

We will show that incremental ISS with inputs taken from a convex and compact set $\mathbb{W} \subseteq \mathbb{R}^m$ can be transformed to uniform global asymptotic stability (UGAS) with respect to the diagonal Δ for an augmented system of (2) via a form of robust incremental feedback. To this end, we define the following function:

$$\mathrm{sat}_{\mathbb{W}}(u) := \begin{cases} u & \text{if } u \in \mathbb{W}, \\ \arg\min_{\nu \in \mathbb{W}} |\nu - u| & \text{if } u \notin \mathbb{W}. \end{cases}$$

For all $w_1, w_2 \in \mathbb{W}$

$$|\mathrm{sat}_{\mathbb{W}}(w_1) - \mathrm{sat}_{\mathbb{W}}(w_2)| = |w_1 - w_2|. \tag{18}$$

Let $\rho \in \mathcal{K}_\infty$. Consider the system

$$\begin{aligned} x_1^+ &= f\big(x_1, \mathrm{sat}_{\mathbb{W}}(d_1 + \rho(|x_1 - x_2|)d_2)\big) \\ x_2^+ &= f\big(x_2, \mathrm{sat}_{\mathbb{W}}(d_1 - \rho(|x_1 - x_2|)d_2)\big) \end{aligned} \tag{19}$$

where $d_1 : \mathbb{Z}_{\geq 0} \to \mathbb{W}$, $d_2 : \mathbb{Z}_{\geq 0} \to [-1, 1]^m$ are viewed as disturbances. Define the sets of possible sequences d_1 and d_2 by $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

With a slight abuse of notation, we denote the solution of system (19) from an initial condition $[\xi_1^T, \xi_2^T]^T \in \mathbb{G}^2$ by $[x_1^T, x_2^T]^T : \mathbb{Z}_{\geq 0} \to \mathbb{G}^2$; i.e., a function that satisfies $[x_1(0, \xi_1, d_1, d_2)^T, x_2(0, \xi_2, d_1, d_2)^T]^T = [\xi_1^T, \xi_2^T]^T$ and

$$\begin{aligned} x_1(k+1, \xi_1, d_1, d_2) &= f\big(x_1(k, \xi_1, d_1, d_2)\big) \\ x_2(k+1, \xi_2, d_1, d_2) &= f\big(x_2(k, \xi_2, d_1, d_2)\big). \end{aligned}$$

UGAS of Δ follows the UGAS definition in [15] (also, [5] for discrete-time systems).

***Definition** 19:* The diagonal Δ is UGAS for system (19) for a fixed $\rho \in \mathcal{K}_\infty$ if the following hold:

(a) (Uniform Stability and Global Boundedness) there exists $\gamma \in \mathcal{K}_\infty$ such that

$$|x_1(k, \xi_1, d_1, d_2) - x_2(k, \xi_2, d_1, d_2)| \leq \gamma(|\xi_1 - \xi_2|) \tag{20}$$

for all $\xi_1, \xi_2 \in \mathbb{G}$, $d_1 \in \mathcal{D}_1$, $d_2 \in \mathcal{D}_2$, and $k \in \mathbb{Z}_{\geq 0}$; and

(b) (Uniform Global Attraction) for each $\delta, \epsilon > 0$, there exists $K = K(\delta, \epsilon) \in \mathbb{Z}_{>0}$ such that

$$\begin{aligned} &|\xi_1 - \xi_2| \leq \delta \quad \Rightarrow \\ &\qquad |x_1(k, \xi_1, d_1, d_2) - x_2(k, \xi_2, d_1, d_2)| \leq \epsilon \end{aligned} \tag{21}$$

for all $\xi_1, \xi_2 \in \mathbb{G}$, $d_1 \in \mathcal{D}_1$, $d_2 \in \mathcal{D}_2$, and $k \in \mathbb{Z}_{\geq K}$.

Now, we are ready to define incremental robust feedback stability.

***Definition** 20:* System (2) is incrementally robustly feedback stable if there exists a class-$\mathcal{K}_\infty$ function ρ (called the incremental stability margin) such that the diagonal Δ is UGAS for system (19).

***Lemma** 21:* Assume the set of inputs $\mathbb{W} \subseteq \mathbb{R}^m$ is compact and convex. If system (2) has incremental asymptotic gain and is incrementally UBIBS, then it is incrementally robustly feedback stable.

The proof of Lemma 21 is omitted due to space constraints.

B.3 A Converse Lyapunov Theorem

In this subsection let $\mathbb{W}$ be compact. We present a converse Lyapunov theorem for incremental ISS. First, we recall a converse result for difference inclusions presented in [5, Theorem 2.7 and Theorem 1.10].

Consider the following difference inclusion

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} \in F\left(\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}\right) \tag{22}$$

where $x_1(k), x_2(k) \in \mathbb{G}$, and the set-valued mapping $F : \mathbb{G}^2 \rightrightarrows \mathbb{G}^2$ is compact, upper semicontinuous, and satisfies

$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \bigcup_{\substack{d_1 \in \mathbb{W} \\ d_2 \in [-1,1]^m}} \begin{bmatrix} f(x_1, \mathrm{sat}_{\mathbb{W}}(d_1 + \rho(|x_1 - x_2|)d_2) \\ f(x_2, \mathrm{sat}_{\mathbb{W}}(d_1 - \rho(|x_1 - x_2|)d_2) \end{bmatrix}. \tag{23}$$

It was shown in [5, Proposition 2.2] that Δ being UGAS for (22) is equivalent to $\mathcal{KL}$-stability for (22) with respect to $(|x_1 - x_2|, |x_1 - x_2|)$; i.e., there exists a function $\beta \in \mathcal{KL}$ such that

$$|x_1(k, \xi_1, d_1, d_2) - x_2(k, \xi_2, d_1, d_2)| \leq \beta(|\xi_1 - \xi_2|, k)$$

for all $\xi_1, \xi_2 \in \mathbb{G}$, $d_1 \in \mathcal{D}_1$, $d_2 \in \mathcal{D}_2$, and $k \in \mathbb{Z}_{\geq 0}$.

By (20), we see that if system (2) is incrementally robustly feedback stable then the difference inclusion (22) is $\mathcal{KL}$-stable with respect to $(|x_1 - x_2|, |x_1 - x_2|)$.

For the $\mathcal{KL}$-stability of the difference inclusion, the converse Lyapunov result [5, Theorem 2.7] states that if the difference inclusion (22) is $\mathcal{KL}$-stable with respect to $(|x_1 - x_2|, |x_1 - x_2|)$ then there exists a smooth Lyapunov function with respect to $(|x_1 - x_2|, |x_1 - x_2|)$; i.e., $V : \mathbb{G}^2 \to \mathbb{R}_{\geq 0}$ such that there exist $\alpha_1, \alpha_2, \alpha \in \mathcal{K}_\infty$ satisfying

$$\alpha_1(|\xi_1 - \xi_2|) \leq V(\xi_1, \xi_2) \leq \alpha_2(|\xi_1 - \xi_2|), \tag{24}$$

$$\begin{aligned} V\Big(&f(x_1, \mathrm{sat}_{\mathbb{W}}(d_1 + \rho(|\xi_1 - \xi_2|)d_2), \\ &\qquad f(\xi_2, \mathrm{sat}_{\mathbb{W}}(d_1 - \rho(|\xi_1 - \xi_2|)d_2)\Big) \\ &- V(\xi_1, \xi_2) \leq -\alpha(V(\xi_1, \xi_2)) \end{aligned} \tag{25}$$

for all $\xi_1, \xi_2 \in \mathbb{G}$, and disturbances $d_1 \in \mathbb{W}$, $d_2 \in [-1, 1]^m$.

***Theorem** 22:* Assume the set of inputs $\mathbb{W} \subseteq \mathbb{R}^m$ is compact and convex. If system (2) is incrementally robustly feedback stable then it admits an implication-form incremental ISS-Lyapunov function.

The proof of Theorem 22 is in Appendix V-C.

***Corollary** 23:* If, in addition to the assumptions of Theorem 22, the set $\mathbb{G}$ is compact, then system (2) is incrementally ISS.

Proof : The proof of Corollary 23 is a direct application of Theorem 15 and Theorem 22. ■

C. A Lyapunov characterization for incremental ISS systems

In this section, we collect the previous results to present a characterization of several incremental ISS related notions.

With Lemmas 13, 18, 21, and Theorems 8, 15, 22 we have the following:

Theorem 24: Suppose that, with inputs from a compact and convex set $\mathbb{W} \subseteq \mathbb{R}^m$, states of (2) evolve in a compact and positively invariant set $\mathbb{G} \subset \mathbb{R}^n$. The following statements are equivalent:

1) System (2) is incrementally ISS.
2) System (2) has an incremental asymptotic gain and is incrementally UBIBS.
3) System (2) is incrementally robustly feeback stable.
4) System (2) admits both implication-form and dissipation-form incremental ISS-Lyapunov functions.

To summarize where the assumptions are required, we note that the implication $(1) \Rightarrow (2)$ does not require any assumption. The implications $(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ require the assumption that the set of inputs $\mathbb{W}$ is compact and convex. Finally, the implication $(4) \Rightarrow (1)$ requires that $\mathbb{W}$ is compact and convex and that $\mathbb{G}$ is compact.

V. Conclusions

We have presented discrete-time versions of incremental stability and incremental ISS proposed in [1] for continuous time systems. In particular, we demonstrated a Lyapunov function characterization and, in addition, a summation-to-summation property of incremental stability. For nonlinear systems with inputs, we presented various equivalent properties to incremental ISS, including an incremental ISS-Lyapunov function characterization. These results are expected to be instrumental for observer analysis, output regulation, and other related problems in the discrete-time setting.

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Appendix

A. Proof of Theorem 5

a. "(1) ⇔ (2)"

We first demonstrate that (5) implies (6). Applying Sontag's Lemma on $\mathcal{KL}$-estimates [13, Proposition 7], given $\beta \in \mathcal{KL}$, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\beta(|\xi_1 - \xi_2|, k) \leq \alpha_2(\alpha_1(|\xi_1 - \xi_2|)e^{-k}) \tag{26}$$

for all $\xi_1, \xi_2 \in \mathbb{G}, k \in \mathbb{Z}_{\geq 0}$. Hence,

$$\alpha_2^{-1}(|x(k, \xi_1) - x(k, \xi_2)|) \leq \alpha_1(|\xi_1 - \xi_2|)e^{-k}. \tag{27}$$

Summing inequality (27) along the solution trajectory,

$$\sum_{\kappa=0}^{k} \alpha_2^{-1}(|x(\kappa, \xi_1) - x(\kappa, \xi_2)|) \leq \sum_{\kappa=0}^{k} \alpha_1(|\xi_1 - \xi_2|)e^{-\kappa}$$
$$\leq \alpha_1(|\xi_1 - \xi_2|)\sum_{\kappa=0}^{\infty} e^{-\kappa} = \alpha_1(|\xi_1 - \xi_2|)\frac{1}{1 - e^{-1}}.$$

Here we note the fact that the inverse of a class-$\mathcal{K}_\infty$ function is also of class-$\mathcal{K}_\infty$. Hence, global incremental AS implies global incremental α-summability.

We now demonstrate that (6) implies (5). To this end, we will show that global incremental α-summability for system (1) implies the diagonal set Δ is globally asymptotically stable (GAS) for system (3). Then, the desired result follows by appealing to [1, Lemma 2.3] which states that the diagonal set Δ is GAS with respect to system (3) if and only if system (1) is globally incrementally AS.

Let $\zeta = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \in \mathbb{G}^2$, $\xi_1, \xi_2 \in \mathbb{G}$, and $z(k, \zeta) := \begin{bmatrix} x_1(k,\xi_1) \\ x_2(k,\xi_2) \end{bmatrix}$. System (1) being globally incrementally α-summable implies

$$\sum_{\kappa=0}^{k} \alpha\big(\sqrt{2}\big|z(\kappa, \zeta)\big|_\Delta\big) \leq \eta\big(\sqrt{2}\big|\zeta\big|_\Delta\big) \tag{28}$$

which, in turn, implies

$$\alpha\big(\sqrt{2}\big|z(k, \zeta)\big|_\Delta\big) \leq \eta\big(\sqrt{2}\big|\zeta\big|_\Delta\big) \tag{29}$$

for all $k \in \mathbb{Z}_{\geq 0}$. Hence, the diagonal Δ is Lyapunov stable for system (3). Furthermore, the series $\sum_{\kappa=0}^{\infty} \alpha(|z(\kappa, \zeta)|_\Delta)$

is bounded. Hence, $|z(\kappa,\zeta)|_\Delta$ converges to zero as k approaches infinity. In other words, the diagonal Δ is GAS for system (3). Therefore, system (1) is globally incrementally AS.

b. "(1) ⇔ (3)"

As mentioned in Remark 2, in the two measurement function framework, incremental stability described in (5) for (1) is equivalent to $\mathcal{KL}$-stability with respect to $(|x_1 - x_2|, |x_1 - x_2|)$ for system (3).

Applying the Lyapunov function characterization [6, Theorem 2] (see also, [5, Theorem 2.7] for the two measurement function case), we see that system (3) is $\mathcal{KL}$-stable with respect to $(|x_1 - x_2|, |x_1 - x_2|)$ if and only if there exist a smooth function $V : \mathbb{G} \times \mathbb{G} \to \mathbb{R}_{\geq 0}$, and functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|\xi_1 - \xi_2|) \leq V(\xi_1, \xi_2) \leq \alpha_2(|\xi_1 - \xi_2|),$$
$$V\Big(f(\xi_1), f(\xi_2)\Big) - V(\xi_1, \xi_2) \leq -\alpha_3(|\xi_1 - \xi_2|)$$

hold for all $\xi_1, \xi_2 \in \mathbb{G}$ and $k \in \mathbb{Z}_{\geq 0}$. Hence, we see that V is an incremental Lyapunov function for (1), which concludes the proof. ■

B. Proof of Theorem 8

The proof follows [2, Lemma 3.5]. Let system (2) admit a dissipation-form incremental ISS-Lyapunov function (10)–(11). The lower bound (10) and inequality (11) imply

$$\begin{aligned} &V(f(\xi_1, w_1), f(\xi_2, w_2)) - V(\xi_1, \xi_2) \\ &\quad \leq -\alpha(V(\xi_1, \xi_2)) + \sigma(|w_1 - w_2|) \end{aligned} \tag{30}$$

where $\alpha \in \mathcal{K}_\infty$ is defined as $\alpha(s) := \alpha_3 \circ \alpha_1^{-1}(s)$ for all $s \in \mathbb{R}_{\geq 0}$.

We denote Id as the identity function; i.e., $\mathrm{Id}(s) = s$ for all $s \in \mathbb{R}_{\geq 0}$. Let ρ be any class-$\mathcal{K}_\infty$ function such that $\mathrm{Id} - \rho \in \mathcal{K}_\infty$. Without loss of generality, assume $\mathrm{Id} - \alpha \in \mathcal{K}$ (see [2, Lemma B.1]).

Fix $w_1, w_2 \in \mathcal{W}$. Define

$$\begin{aligned} \mathbb{S} := \Big\{&(\eta_1, \eta_2) : \eta_1, \eta_2 \in \mathbb{G}, \\ &V(\eta_1, \eta_2) \leq \alpha^{-1} \circ \rho^{-1} \circ \sigma(\|w_1 - w_2\|_\infty)\Big\}. \end{aligned} \tag{31}$$

Claim 2: The set $\mathbb{S}$ is forward invariant. In other words, if $(\xi_1, \xi_2) \in \mathbb{S}$ then $(x(k, \xi_1, w_1), x(k, \xi_2, w_2)) \in \mathbb{S}$ for all $k \in \mathbb{Z}_{\geq 0}$.

Proof of Claim 2: Let $(\xi_1, \xi_2) \in \mathbb{S}$. Let $w_1(0), w_2(0) \in \mathbb{W}$ be the first elements of w_1, w_2, respectively. Then $V(\xi_1, \xi_2) \leq \alpha^{-1} \circ \rho^{-1} \circ \sigma(\|w_1 - w_2\|_\infty)$ and, together with (30), we have

$$\begin{aligned} &V(f(\xi_1, w_1(0)), f(\xi_2, w_2(0))) \\ &\leq V(\xi_1, \xi_2) - \alpha(V(\xi_1, \xi_2)) + \sigma(\|w_1 - w_2\|_\infty) \\ &= (\mathrm{Id} - \alpha)(V(\xi_1, \xi_2)) + \sigma(\|w_1 - w_2\|_\infty) \\ &\leq (\mathrm{Id} - \alpha) \circ \alpha^{-1} \circ \rho^{-1} \circ \sigma(\|w_1 - w_2\|_\infty) \\ &\quad + \sigma(\|w_1 - w_2\|_\infty) \\ &= \alpha^{-1} \circ \rho^{-1} \circ \sigma(\|w_1 - w_2\|_\infty) \\ &\quad - \rho^{-1} \circ \sigma(\|w_1 - w_2\|_\infty) + \sigma(\|w_1 - w_2\|_\infty). \end{aligned} \tag{32}$$

Observe that, since $\mathrm{Id} - \rho \in \mathcal{K}_\infty$,

$$\begin{aligned} &-\rho^{-1} \circ \sigma(\|w_1 - w_2\|_\infty) + \sigma(\|w_1 - w_2\|_\infty) \\ &\quad = -(\mathrm{Id} - \rho) \circ \rho^{-1} \circ \sigma(\|w_1 - w_2\|_\infty) \leq 0. \end{aligned} \tag{33}$$

Then, following (32)-(33), the claim is proved since

$$\begin{aligned} &V(f(\xi_1, w_1(0)), f(\xi_2, w_2(0))) \\ &\quad \leq \alpha^{-1} \circ \rho^{-1} \circ \sigma(\|w_1 - w_2\|_\infty) \end{aligned}$$

which implies $(f(\xi_1, w_1(0)), f(\xi_2, w_2(0))) \in \mathbb{S}$ given $(\xi_1, \xi_2) \in \mathbb{S}$ for the fixed $w_1, w_2 \in \mathcal{W}$. ■

For $(\xi_1, \xi_2) \in \mathbb{S}$, applying (10) yields

$$\begin{aligned} &\alpha_1(|x(k, \xi_1, w_1) - x(k, \xi_2, w_2)|) \\ &\quad \leq V(x(k, \xi_1, w_1), x(k, \xi_2, w_2)) \\ &\quad \leq \alpha^{-1} \circ \rho^{-1} \circ \sigma(\|w_1 - w_2\|_\infty). \end{aligned} \tag{34}$$

Therefore,

$$\begin{aligned} &|x(k, \xi_1, w_1) - x(k, \xi_2, w_2)| \\ &\leq \alpha_1^{-1} \circ \alpha^{-1} \circ \rho^{-1} \circ \sigma(\|w_1 - w_2\|_\infty) =: \gamma(\|w_1 - w_2\|_\infty) \end{aligned}$$

where $\gamma \in \mathcal{K}_\infty$ is defined as $\gamma(s) := \alpha_1^{-1} \circ \alpha^{-1} \circ \rho^{-1} \circ \sigma(s)$ for all $s \in \mathbb{R}_{\geq 0}$.

For $(\xi_1, \xi_2) \notin \mathbb{S}$, (31) implies $\rho \circ \alpha(V(\xi_1, \xi_2)) > \sigma(\|w_1 - w_2\|_\infty)$. As a consequence, (30) results in

$$\begin{aligned} &V(f(\xi_1, w_1(0)), f(\xi_2, w_2(0))) - V(\xi_1, \xi_2) \\ &\quad \leq -\alpha(V(\xi_1, \xi_2)) + \rho \circ \alpha(V(\xi_1, \xi_2)) \\ &\quad \leq -(\mathrm{Id} - \rho) \circ \alpha(V(\xi_1, \xi_2)). \end{aligned} \tag{35}$$

By a standard comparison Lemma (see [3, Lemma 4.3]), there exists $\tilde{\beta} \in \mathcal{KL}$ such that

$$\begin{aligned} &\alpha_1(|x(k, \xi_1, w_1) - x(k, \xi_2, w_2)|) \\ &\quad \leq V(x(k, \xi_1, w_1), x(k, \xi_2, w_2)) \leq \tilde{\beta}(V(\xi_1, \xi_2), k) \\ &\quad \leq \tilde{\beta}(\alpha_2(|\xi_1 - \xi_2|), k). \end{aligned} \tag{36}$$

Define $\beta \in \mathcal{KL}$ by $\beta(r, s) := \alpha_1^{-1} \circ \tilde{\beta}(\alpha_2(r), s)$ for all $r, s \in \mathbb{R}_{\geq 0}$. Then, combining (35) and (36) yields

$$\begin{aligned} &|x(k, \xi_1, w_1) - x(k, \xi_2, w_2)| \\ &\quad \leq \max\{\beta(|\xi_1 - \xi_2|, k), \gamma(\|w_1 - w_2\|_\infty)\} \\ &\quad \leq \beta(|\xi_1 - \xi_2|, k) + \gamma(\|w_1 - w_2\|_\infty). \end{aligned}$$

Therefore, system (2) is incrementally ISS. ■

C. Proof of Theorem 22

In order to derive a Lyapunov function for system (2), let $w_1 = \mathrm{sat}_{\mathbb{W}}(d_1 + \rho(|\xi_1 - \xi_2|)d_2), w_2 = \mathrm{sat}_{\mathbb{W}}(d_1 - \rho(|\xi_1 - \xi_2|)d_2)$. Then, by the convexity of $\mathbb{W}$, we see that

$$|w_1 - w_2| \leq 2|\rho(|\xi_1 - \xi_2|)d_2| \leq 2\rho(|\xi_1 - \xi_2|). \tag{37}$$

Hence, we can rewrite (25) as

$$\begin{aligned} &\chi(|\xi_1 - \xi_2|) \geq |w_1 - w_2| \\ &\Rightarrow V(f(\xi_1, w_2), f(\xi_2, w_2)) - V(\xi_1, \xi_2) \leq -\alpha(V(\xi_1, \xi_2)) \end{aligned}$$

where $\chi \in \mathcal{K}_\infty$ is defined as $\chi(s) = 2\rho(s)$ for all $s \in \mathbb{R}_{\geq 0}$.

Therefore, we conclude that V is an implication-form incremental ISS-Lyapunov function for system (2). ■