Discrete time monotone systems: Criteria for global asymptotic stability and applications

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***Abstract*— For two classes of monotone maps on the $\mathbf{n}$-dimensional positive orthant we show that for a discrete dynamical system induced by a map the origin of $\mathbb{R}^n_+$ is globally asymptotically stable, if and only if the map Γ is such that for any point in $\mathbf{s} \in \mathbb{R}^n_+$, $\mathbf{s} \neq \mathbf{0}$, the image-vector $\Gamma(\mathbf{s})$ is such that at least one component is strictly less than the corresponding component of $\mathbf{s}$.**

One class is the set of $\mathbf{n} \times \mathbf{n}$ matrices of class $\mathcal{K}_\infty$ functions; these induce monotone operators on $\mathbb{R}^n_+$. Maps of the other class satisfy some geometric property for an invariant set.

***Keywords*— monotone maps, spectral radius, one component decrease condition, global asymptotic stability**

I. Introduction

We consider monotone (that is order preserving) maps which map the nonnegative orthant $\mathbb{R}^n_+$ of the n-dimensional Euclidean space $\mathbb{R}^n$ into itself. Such a map $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ gives rise to an associated dynamical system defined by

$$s(k+1) = \Gamma(s(k)), \quad \text{for } k = 1, 2, \dots \tag{1}$$

and $s(0) = s_0 \in \mathbb{R}^n_+$. We endow $\mathbb{R}^n_+$ with the standard partial order $\geq$ defined by

$$x \geq y \iff x - y \in \mathbb{R}^n_+. \tag{2}$$

If Γ is linear, the stability condition that the spectral radius of Γ is less than 1 is equivalent to the operator inequality

$$\text{there exists no } s \neq 0, \text{ such that } \Gamma(s) \geq (s); \tag{3}$$

note that the latter condition is also meaningful for nonlinear Γ. We call monotone operators Γ satisfying (3) *nowhere increasing* (on $\mathbb{R}^n_+$).

The concern of this paper are the relations between property (3) and global asymptotic stability of (1); for finite dimensional linear Γ these are of course equivalent.

Monotone maps satisfying (3) arise in the context of large-scale interconnections of input-to-state stable (ISS) subsystems. Here the map Γ arises as a matrix whose entries are strictly increasing functions (class $\mathcal{K}_\infty$-functions), which describe the interconnection gains between ISS subsystems. Recently the authors proved in [4] that if there exists a 'robustness' operator D, such $D \circ \Gamma$ is nowhere increasing, then the large-scale interconnection system is also ISS.

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The first small-gain theorem for the feedback interconnection of two ISS systems was given by Jiang, Teel and Praly in [8]. Since then many more ISS small-gain type theorems followed, for references and discussion see [4].

Monotone maps induced by matrices of class $\mathcal{K}$ functions and their dynamics are of course a very special case. The theory of more general monotone maps and induced dynamics is still an active field of research, see for example [11] or [12]. In [6] Hirsch and Smith give a state of the art overview on discrete-time monotone dynamics. A work related to this article is [1], where Angeli and Sontag present results on monotone systems that were inspired by questions in molecular biology modeling. In [2] they introduce signed graphs for monotone maps, which in the case of matrices with entries in $(\mathcal{K}_\infty \cup \{0\})$ agree with the graphs that we associate to such maps.

We are going to investigate the relations between property (3) and global asymptotic stability of the origin (0-GAS) of (1) for general monotone maps $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$. Using only monotonicity we establish (3) as a consequence of 0-GAS of (1) in Proposition 5.2, while the converse implication is true for matrices with entries in $(\mathcal{K}_\infty \cup \{0\})$ if we add a 'robustness term' in (3), see Theorem 7.10. In two dimensions we can swap the matrix structure for geometrical properties of an invariant set to obtain a similar result (Proposition 6.3).

A rigorous problem statement will be given in Section II. Notational foundations and some remarks on class $\mathcal{K}_\infty$-functions follow in Sections III and IV. Results requiring only monotonicity, continuity, and/or condition (3) are stated in Section V, while Section VI deals with the two dimensional case. The theory for matrices with entries in $(\mathcal{K}_\infty \cup \{0\})$ is presented in Section VII. In this case the special structure allows for stronger statements. Consequences for large-scale 'small-gain' interconnections of ISS systems are given in SectionVIII.

II. Problem statement

For a monotone map $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, such that $\Gamma(0) = 0$, we are interested in the relations between the following two properties:

1) $\Gamma \not\geq \mathrm{id}$, that is, by definition for all $x \in \mathbb{R}^n_+$, $x \neq 0$, we have $\Gamma(x) \not\geq x$; and
2) the discrete system Σ defined by

$$\Sigma : x(0) \in \mathbb{R}^n_+, \ x(k+1) := \Gamma(x(k)), \ k \in \mathbb{N}_0, \tag{4}$$

is globally asymptotically stable in 0 (0-GAS), i.e.,

a) (Stability) For every $\varepsilon > 0$ there exists a $\delta > 0$, such that whenever $\|x(0)\| < \delta$, then $\|x(k)\| < \varepsilon$ for all $k \in \mathbb{N}$.
b) (Attractivity) $\forall x \in \mathbb{R}^n_+ : \Gamma^k(x) \to 0$ as $k \to \infty$.

Note that so far we did not make any continuity assumptions whatsoever. We will provide the following answers: Proposition 5.2 states that 2b) implies 1. Lemma 6.3 gives the converse implication and even stability for $n = 2$ for continuous Γ, if in addition for a diagonal robustness operator D we have $D \circ \Gamma \not\geq \mathrm{id}$ and some constraints on the geometry of the set Ψ^Γ, which is to be defined in equation (6), are satisfied. If Γ is a matrix with entries in $(\mathcal{K}_\infty \cup \{0\})$ the same is proved without the restriction on Ψ^Γ in Theorem 7.10. Finally, in the context of large-scale networks of ISS systems this gives a sufficient condition for the input-to-state stability (ISS) of the interconnected system, see Theorem 8.1.

III. Notation

A. Numbers, ordering

By $\mathbb{N}$ we denote the positive integers, by $\mathbb{N}_0$ the union $\mathbb{N} \cup \{0\}$, by $\mathbb{R}$ the real numbers, by $\mathbb{R}_+$ the nonnegative real numbers, and by $\mathbb{R}^n_+ = \mathbb{R}_+ \times \ldots \times \mathbb{R}_+$ the nonnegative orthant of $\mathbb{R}^n$. On the latter we have a partial ordering, which is induced by the componentwise ordering on $\mathbb{R}$: For $x, y \in \mathbb{R}^n$ the relation $x \leq y$ ($x < y$) is defined by $x_i \leq y_i$ ($x_i < y_i$) for all $i = 1, \ldots, n$. While the definition of $\leq$ agrees with the one in (2), it is important to note that the meaning of $x < y$ does not coincide with the usual meaning, $x \geq y$ and $x \neq y$. Consequently, the notation $x \not\geq y$ means that there exists an index i, such that $x_i < y_i$. For maps $A, B : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ we define

$$A \not\geq B$$

as a point wise relation with the exception of the origin; that is, there exists no $x \in \mathbb{R}^n_+$, $x \neq 0$, such that $A(x) \geq B(x)$. The map $A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is *monotone*, if for $x, y \in \mathbb{R}^n_+$ such that $x \leq y$ we have $A(x) \leq A(y)$. A monotone map $A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is *nowhere increasing* if $A \not\geq \mathrm{id}$.

B. Comparison functions

Recall that a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$, or $\rho \in \mathcal{K}$, if $\rho(0) = 0$, ρ is strictly increasing and continuous. If, in addition, ρ is unbounded, then we say ρ is of class $\mathcal{K}_\infty$ (or, $\rho \in \mathcal{K}_\infty$). Sometimes we enrich these spaces by the function $0 : \mathbb{R}_+ \to \mathbb{R}_+$ mapping everything to 0, which we denote by $\mathcal{K} \cup \{0\}$ or $\mathcal{K}_\infty \cup \{0\}$. A function $\beta : \mathbb{R}^2_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if it is of class $\mathcal{K}$ in the first argument and, whenever the first argument is fixed, it is decreasing in the second argument with limit zero.

C. Matrices of $\mathcal{K}$-functions, ordering for monotone maps

By $(\mathcal{K} \cup \{0\})^{n \times n}$ we denote the set of $n \times n$-matrices with elements in $\mathcal{K} \cup \{0\}$. Given a matrix $\Gamma \in (\mathcal{K} \cup \{0\})^{n \times n}$ and a vector $s \in \mathbb{R}^n_+$ we define the vector $\Gamma(s) \in \mathbb{R}^n_+$ by

$$\Gamma(s)_i := \sum_{j=1}^n \gamma_{ij}(s_j) \quad \text{for } i = 1, \ldots, n.$$

Note, that if all γ_{ij} are linear, then this definition is compatible with the usual matrix vector multiplication from linear algebra.

D. Projections

On $\mathbb{R}^n_+$ for any index set $\emptyset \neq I \subset \{1, \ldots, n\}$ we denote by $P_I : \mathbb{R}^n_+ \to \mathbb{R}^{\#I}_+$ the projection of the coordinates in $\mathbb{R}^n_+$ corresponding to the indices in I onto $\mathbb{R}^{\#I}$.

The corresponding injection is $S_I : \mathbb{R}^{\#I}_+ \to \mathbb{R}^n_+$ mapping $x \mapsto (x_1 e_{i_1} + \ldots + x_{\#I} e_{i_{\#I}})$, where we write $I = \{i_1, \ldots, i_{\#I}\}$ and denote by $(e_k)_{k=1,\ldots,n}$ the standard basis of $\mathbb{R}^n_+$.

For any index set $\emptyset \neq I \subset \{1, \ldots, n\}$ and vector $x \in \mathbb{R}^n_+$ denote by $x|_I$ the vector in $\mathbb{R}^n_+$ with elements

$$(x|_I)_i = \begin{cases} x_i & \text{if } i \in I \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

E. Graphs, adjacency matrix

With an $n \times n$ matrix with entries in $(\mathcal{K}_\infty \cup \{0\})$ we associate a (directed) graph $G = (V, E)$ with vertices $V = \{1, \ldots, n\}$ and edges $E \subset V \times V$, which consists of all ordered pairs $(i, j) \in V \times V$ such that $\gamma_{ij} \not\equiv 0$. The *adjacency matrix* of a graph $G = (V, E)$ with $V = \{1, \ldots, n\}$ and $E \subset V \times V$ is the matrix $A_G := (a_{ij})$ with

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \text{ and} \\ 0 & \text{else.} \end{cases}$$

Given a $\rho \in \mathcal{K}_\infty$, we define a matrix $D := \mathrm{diag}_n(\mathrm{id} + \rho) := (\delta_{ij} \cdot (\mathrm{id} + \rho))_{n \times n}$, where δ_{ij} denotes the Kronecker symbol,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else.} \end{cases}$$

A nonnegative $n \times n$ matrix A is *irreducible*, if for every $i, j \in \{1, \ldots, n\}$ there is a $k \in \mathbb{N}$ such that the (i, j)-th entry of A^k is positive, which is denoted by $a^{(k)}_{ij} > 0$. This can be stated equivalently as that the graph of A is strongly connected, i.e., from any vertex there is a path to any other vertex. If the number k can be chosen independently of (i, j), then A is *primitive*. If A is not irreducible, then it is *reducible*.

In Section VII we will need the graph concept also for powers of Γ. To this end let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$ and consider $\Gamma^k : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ for some $k \geq 1$. Clearly, the map Γ^k is monotone, continuous and satisfies $\Gamma^k(0) = 0$. With

Γ^k we associate the graph $G = G(\Gamma^k) = (V, E)$, with vertices $V = \{1, \dots, n\}$ and edges

$$\begin{aligned} E &= \big\{(i,j) \in V \times V : \forall x \in \mathbb{R}^n_+ : \\ &\qquad t \mapsto P_j(\Gamma(x + t \cdot e_i)), \\ &\qquad t \in \mathbb{R}_+ \text{ is unbounded}\big\} \\ &= \big\{(i,j) \in V \times V : t \mapsto P_j(\Gamma(t \cdot e_i)), \\ &\qquad t \in \mathbb{R}_+, \text{ is of class } \mathcal{K}_\infty\big\}. \end{aligned}$$

For notational simplicity we just say that a map Γ has some of the properties as (ir-)reducibility/primitivity, if the adjacency matrix A_G of the graph $G = G(\Gamma)$ does have this property. In the sequel we sometimes refer to A_{G^k} as the adjacency matrix of the graph of Γ^k.

IV. Some remarks on $\mathcal{K}$ and $\mathcal{K}_\infty$ functions

In [4], as well as in the following sections, the notion $D \circ \Gamma$ is used, D being a diagonal matrix with entries in $\mathcal{K}_\infty$, Γ a monotone map. Proposition 4.3 states equivalent formulations of this notion that will be useful in the proof of Lemma 6.1, but may be of independent interest. We begin with two technical observations.

Lemma 4.1: For any $\rho \in \mathcal{K}_\infty$, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, such that

$$(\rho + \text{id}) = (\alpha_1 + \text{id}) \circ (\alpha_2 + \text{id}).$$

Proof: Choose, e.g., $\alpha_2 = \frac{1}{2}\rho$ and $\alpha_1 = \frac{1}{2}\rho \circ (\frac{1}{2}\rho + id)^{-1}$. Then

$$\begin{aligned} (\alpha_1 + \text{id}) \circ (\alpha_2 + \text{id}) = \alpha_1(\alpha_2 + \text{id}) + \alpha_2 + \text{id} = \\ \frac{1}{2}\rho \circ (\frac{1}{2}\rho + id)^{-1} \circ (\frac{1}{2}\rho + \text{id}) + \frac{1}{2}\rho + id = \\ (\frac{1}{2}\rho \circ \text{id} + \frac{1}{2}\rho + \text{id}) = \rho + \text{id}. \end{aligned}$$

■

This immediately extends to diagonal operators.

Lemma 4.2: For every $D = \text{diag}_n(\text{id} + \rho), \rho \in \mathcal{K}_\infty$ there exist $\tilde{D} = \text{diag}_n(\text{id} + \alpha_1)$ and $\hat{D} = \text{diag}_n(\text{id} + \alpha_2), \alpha_i \in \mathcal{K}_\infty, i = 1, 2$, such that

$$D = \tilde{D} \circ \hat{D}.$$

Note, as a consequence, for D as in Lemma 4.2, there exists a $\bar{D} = \text{diag}_n(\text{id} + \bar{\rho})$, such that $D \geq \bar{D} \circ \bar{D}$. This class $\mathcal{K}_\infty$ function is defined by $\bar{\rho}(t) = \min\{\rho_1(t), \rho_2(t)\}$ for $t \geq 0$. So in Proposition 4.3 without loss of generality we may even assume that $D_1 = D_2$ (using monotonicity of Γ):

Proposition 4.3: For $\Gamma \in (\mathcal{K} \cup \{0\})^{n \times n}$ the following are equivalent:

(i) $\exists \rho \in \mathcal{K}_\infty, D = \text{diag}_n(\text{id} + \rho) : \Gamma \circ D \ngeq \text{id}$,
(ii) $\exists \rho \in \mathcal{K}_\infty, D = \text{diag}_n(\text{id} + \rho) : D \circ \Gamma \ngeq \text{id}$,
(iii) $\exists \rho_1, \rho_2 \in \mathcal{K}_\infty, D_1 = \text{diag}_n(\text{id} + \rho_1)$, $D_2 = \text{diag}_n(\text{id} + \rho_2) : D_1 \circ \Gamma \circ D_2 \ngeq \text{id}$.

The easy equivalence transformations are omitted.

V. Exploiting monotonicity

In this section we investigate the problem stated in Section II for maps Γ that are only monotone. For some results we also need continuity of Γ. A first general result is Proposition 5.2

Lemma 5.1: Let $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone, $\Gamma(0) = 0$. Assume $\Gamma \ngeq \text{id}$. Then for any index set I such that $0 \neq I \subset \{1, \dots, n\}$, we have

$$P_I \circ \Gamma \circ S_I \ngeq \text{id}_{\mathbb{R}^{\#I}_+}.$$

Proof: The easy proof is left to the reader. ■

One of the highlights of this section, though not difficult to prove, is the following statement, that terms of the problem of Section II for general Γ already property 2b implies property 1:

Proposition 5.2: Let $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone, $\Gamma(0) = 0$. Then attractivity of $x^* = 0$ for the associated discrete dynamical system implies $\Gamma \ngeq \text{id}$.

Proof: We argue indirectly: Suppose there exists an $x \in \mathbb{R}^n_+, x \neq 0$ such that $\Gamma(x) \geq x$. By monotonicity of Γ we have $\Gamma^2(x) \geq \Gamma(x) (\geq x)$ and inductively $\Gamma^k(x) \geq x \neq 0$. Hence $\Gamma^k(x) \not\to 0$ as $k \to \infty$, contradicting 2b, i.e., $\lim_{k\to\infty} \Gamma^k(x) = 0$. ■

The following result will prove as powerful tool in the indirect proofs of several results to follow.

Lemma 5.3: Let $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone, $\Gamma(0) = 0$. Then $\Gamma \ngeq \text{id}$ implies $\Gamma^k \ngeq \text{id}$ for all $k \in \mathbb{N}$.

Proof: We argue indirectly: Suppose there exists an $x \in \mathbb{R}^n_+, x \neq 0$ and a $k > 0$, such that $\Gamma^k(x) \geq x$. Define $z := \max_{l=0,\dots,k-1} \Gamma^l(x)$. Since the case $l = 0$ is included we have $z \neq 0$ and by monotonicity $z \geq 0$. We obtain

$$\begin{aligned} \Gamma(z) \geq \max_{l=1,\dots,k} \Gamma^l(x) &= \max_{l=0,\dots,k} \Gamma^l(x) \\ &\geq \max_{l=0,\dots,k-1} \Gamma^l(x) = z, \end{aligned}$$

which in turn contradicts $\Gamma \ngeq \text{id}$. Hence the lemma is proved. ■

If in addition Γ is continuous then already boundedness of a trajectory of system (1) implies that this trajectory converges to zero; this fact will be used frequently in the following.

Lemma 5.4: Let $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone and continuous, $\Gamma(0) = 0$. For fixed $x \in \mathbb{R}^n_+$ let $\{\Gamma^k(x)\}_{k=0,1,2,\dots}$ be bounded and let $\Gamma \ngeq \text{id}$. Then $\Gamma^k(x) \to 0$ as $k \to \infty$.

Proof: Consider the ω-limit set of x

$$\begin{aligned} \omega(x) := \{y : \exists \text{ subsequence } \{k_j\}_{j=1,2,\dots} \\ \text{s.t. } \Gamma^{k_j}(x) \to y \text{ as } j \to \infty\}. \end{aligned}$$

Since any bounded sequence in $\mathbb{R}^n$ contains a convergent subsequence $\omega(x)$ is not empty.

Note that by continuity of Γ the set $\omega(x)$ is invariant under Γ: For any $y \in \omega(x)$ the image $\Gamma(y)$ is also in $\omega(x)$ and there exists a preimage $z \in \omega(x)$ such that $\Gamma(z) = y$.

We define $z := \sup \omega(x)$ which is finite. For every $y \in \omega(x)$ we have $z \geq y$ and hence $\Gamma(z) \geq \Gamma(y)$. By invariance this yields $\Gamma(z) \geq \sup\{\Gamma(y) : y \in \omega(x)\} = z$. But this contradicts $\Gamma \not\geq \mathrm{id}$ if $z \neq 0$. Hence $\omega(x) = \{0\}$. ∎

In the following we will occasionally make use of properties of the sets defined below. Suppose an operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is given. Then define

$$\Psi_i^\Gamma := \{s \in \mathbb{R}^n_+ : (\Gamma(s))_i \leq s_i\} \tag{5}$$

$$\Psi^\Gamma := \bigcap_{i=1}^n \Psi_i^\Gamma = \{s \in \mathbb{R}^n_+ : \Gamma(s) \leq s\} \tag{6}$$

$$\Omega_i^\Gamma := \{s \in \mathbb{R}^n_+ : (\Gamma(s))_i < s_i\} \tag{7}$$

$$\Omega^\Gamma := \bigcap_{i=1}^n \Omega_i^\Gamma = \{s \in \mathbb{R}^n_+ : \Gamma(s) < s\}. \tag{8}$$

If there is no ambiguity regarding the operator Γ, then we will omit the north-east index Γ. Obviously we have $\Omega_i \subset \Psi_i$, $i = 1, \ldots, n$.

Lemma 5.5: Let $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone and continuous, $\Gamma(0) = 0$. Then the following are equivalent:

(i) $\Gamma \not\geq \mathrm{id}$
(ii) $\bigcup_{i=1}^n \Omega_i = \mathbb{R}^n_+ \setminus \{0\}$.

Proof: This is easily seen, so the proof is left to the reader. ∎

The next result has interesting consequences: Under mild assumptions there are points $x \in \mathbb{R}^n_+$ arbitrarily far away from the origin, such that for every initial value $s_0 \in \mathbb{R}^n_+$ with $s_0 \geq x$ for system (1) the corresponding trajectory is attracted to 0.

Proposition 5.6: Let $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone and continuous, $\Gamma(0) = 0$. Then $\Gamma \not\geq \mathrm{id}$ implies $\Omega \cap S_r \neq \emptyset$ for all $r > 0$, where S_r denotes the sphere around the origin in $\mathbb{R}^n_+$ of radius $r > 0$ with respect to the 1-norm, $S_r = \{s \in \mathbb{R}^n_+ : \sum_{i=1}^n s_i = r\}$.

For the proof of this proposition we need a famous result, that we state here for the convenience of the reader:

Theorem 5.7 (Knaster-Kuratowski-Mazurkiewicz, 1929): Let Δ_n denote unit n-simplex, and for a face σ of Δ_n let $\sigma^{(0)}$ denote the set of vertices of σ.

If a family $\{A_i | i \in \Delta_n^{(0)}\}$ of subsets of Δ_n is such that all the sets are closed or all are open, and each face σ of Δ_n is contained in the corresponding union $\bigcup\{A_i | i \in \sigma^{(0)}\}$, then there is a point common to all the sets.

Proof: The original proof for closed sets was given in [9], while the formulation above is taken from [7] and was proved in [10]. ∎

Proof: [Proof of Proposition 5.6] Note that S_r for $r > 0$ is a simplex with vertices $r \cdot e_i$, $i = 1, \ldots, n$. Each (nonempty) face spanned by $r \cdot e_i$, $i \in I \subset \{1, \ldots, n\}$, fulfills the assumptions of the Knaster-Kuratowski-Mazurkewicz theorem, i.e., it is contained in the union $\bigcup_I (\Omega_i \cap S_r)$. Then the KKM-theorem implies that $\bigcap_1^n (\Omega_i \cap S_r) \neq \emptyset$. ∎

Lemma 5.8: Let $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone, $\Gamma(0) = 0$, $\Gamma \not\geq \mathrm{id}$. Then each trajectory of Σ given by (4) starting in Ψ is bounded.

Proof: This follows easily by monotonicity of Γ, since $x_0 \in \Psi$ implies $\Gamma(x_0) \leq x_0$ and iterated application of Γ gives

$$0 \leq x(k) = \Gamma^k(x_0) \leq \Gamma^{k-1}(x_0) \leq \ldots \leq \Gamma(x_0) \leq x_0$$

for all $k \geq 1$. ∎

As a consequence we have the following intermediate result, that is also used in the companion paper [5].

Proposition 5.9: Let $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone and continuous, $\Gamma(0) = 0$, $\Gamma \not\geq \mathrm{id}$. Then every trajectory of system (1) starting in Ψ^Γ is attracted to 0.

Proof: This follows from Lemma 5.8 and Lemma 5.4. ∎

So far we only considered attractivity of the origin of $\mathbb{R}^n_+$, but neglected stability. As it turns out, the latter is a consequence of the first:

Lemma 5.10: Let $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone and continuous, $\Gamma(0) = 0$ and $\Gamma \not\geq \mathrm{id}$. If every trajectory of system (4) is attracted to the origin, then the origin is also stable for system (4).

Proof: By Proposition 5.6 and given $\varepsilon > 0$ we may choose a $y \in \bigcap_{i=1}^n \Omega_i \cap S_\varepsilon$, where S_ε denotes the sphere of radius ε in $\mathbb{R}^n_+$ with respect to the 1-norm. Define $\delta > 0$, $\delta \in \mathbb{R}$, by

$$\delta := \sup\{d \in \mathbb{R}, d > 0 : x < y \ \forall x \in B_d(0) \cap \mathbb{R}^n_+\},$$

where $B_d(0)$ denotes the open ball of radius d with respect to the 1-norm around the origin. Clearly for $|x| < \delta$ we have $x < y$. By the choice of y we have $\Gamma(y) < y$, and hence $y > \Gamma(y) \geq \Gamma^2(y) \geq \ldots$ The same applies for $|x| < \delta$, hence $|\Gamma^k(x)| < \varepsilon$, for all $k \geq 0$, which proves stability. ∎

As a consequence of Proposition 5.9 we obtain:

Corollary 5.11: Let $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone and continuous, $\Gamma(0) = 0$, $\Gamma \not\geq \mathrm{id}$. If for a initial value $s_0 \in \mathbb{R}^n_0$ of system (1) there exists a $k \geq 0$ and some $y \in \Psi$, such that $s(k) \leq y$, then the corresponding trajectory converges to 0.

Another easy but important consequence is the next statement, that relies on the two preceding results: If the set Ψ is unbounded in every component, then (1) is 0-GAS.

Proposition 5.12: Let $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone and continuous, $\Gamma(0) = 0$, $\Gamma \not\geq \mathrm{id}$. If for every $x \in \mathbb{R}^n_+$ there exists a $y \in \Psi$, such that $y \geq x$, then the origin of $\mathbb{R}^n_+$ is GAS for system (1).

Proof: For every initial value s_0 of (1), there exists a $y \in \Psi$ such that $y \geq s_0$. By monotonicity of Γ we have $\Gamma^k(s_0) \leq \Gamma^k(y)$ for all $k \geq 0$. The remainder follows by application of Proposition 5.9 and Lemma 5.10. ∎

For completeness, we state the by now obvious result: If Γ is bounded, then we immediately get 0-GAS of (1).

Lemma 5.13: Let $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone, continuous, and bounded, such that $\Gamma \not\geq \mathrm{id}$. Then system (1) is 0-GAS.

Proof: Note that $\Gamma(0) = 0$ is a consequence of $\Gamma \not\geq \mathrm{id}$ and continuity. Now apply Lemma 5.4 and Lemma 5.10. ∎

VI. Two dimensional case

For Γ a $n \times n$ matrix with entries in $(\mathcal{K}_\infty \cup \{0\})$ it has been shown in [4] that

$$D \circ \Gamma \not\geq \mathrm{id} \quad \text{for some } D = \mathrm{diag}_n(\mathrm{id} + \rho), \tag{9}$$

where $\rho \in \mathcal{K}_\infty$, is sufficient to deduce that (1) is 0-GAS.

In the case $n = 2$ we provide a dichotomy for more general Γ, such that if (9) holds, then for a subsequence at least one component of each trajectory of (1) converges to 0:

Lemma 6.1: Let $\Gamma : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ be monotone and continuous, $\Gamma(0) = 0$. Suppose there exists an operator $D : \mathbb{R}^2_+ \to \mathbb{R}^2_+$, $D = \mathrm{diag}_2(\mathrm{id} + \rho)$, $\rho \in \mathcal{K}_\infty$, such that $D \circ \Gamma \not\geq \mathrm{id}$. Suppose further, that for some $x_0 \in \mathbb{R}^2_+$ the orbit $O(x_0) = \{\Gamma^k(x)\}_{k=0,1,2,\dots}$ is unbounded. Then there exists a subsequence $\{k_j\}_{j \in \mathbb{N}_0}$ such that as $j \to \infty$ exactly one of the following holds:

(i) $(\Gamma^{k_j}(x_0))_1 \nearrow \infty$ and $(\Gamma^{k_j}(x_0))_2 \searrow 0$,
(ii) $(\Gamma^{k_j}(x_0))_1 \searrow 0$ and $(\Gamma^{k_j}(x_0))_2 \nearrow \infty$.

Proof: First note, that by assumption and Lemma 4.2 there exist operators $D_i = \mathrm{diag}_2(\mathrm{id} + \rho_i), \rho_i \in \mathcal{K}_\infty, i = 1,2$, such that $D_1 \circ D_2 = D$ and therefore $D_1 \circ \Gamma \circ D_2 \not\geq \mathrm{id}$. By Lemma 5.3 we even have $D_1 \circ \Gamma^L \circ D_2 \not\geq \mathrm{id}$ for all $L \geq 1$.

Since $O(x_0)$ is unbounded, necessarily one component must be unbounded, and without loss of generality that is the first component. So we have to show that $(\Gamma^{k_j}(x_0))_2 \searrow 0$ as $j \to \infty$.

We may pick a subsequence $\{k_j\}_{j=0}^\infty$ such that the first component of

$$(x_1(k_j), x_2(k_j))^T := x(k_j) := \Gamma^{k_j}(x_0)$$

is strictly monotone and unbounded. The sequence $x_2(k_j)$ must be strictly decreasing, since otherwise we would have $x(k_{j+1}) \geq x(k_j)$, but we know $x(k_{j+1}) \not\geq x(k_j)$.

It remains to prove that $x_2(k_j)$ decreases to zero. So suppose there exists a constant $C > 0$ such that $x_2(k_j) \searrow C$. Then for $\varepsilon \in]0, \rho_1(C)[$ there exist $j' \in \mathbb{N}$ and $\varepsilon' < \varepsilon$ such that

$$x(k_{j'}) = \begin{bmatrix} x_1(k_{j'}) \\ C + \varepsilon' \end{bmatrix} > 0.$$

Now for $l \geq 1$ we have

$$\begin{aligned} x(k_{j'+l}) &= \Gamma^L(x(k_{j'})) \quad \text{for some } L = L(l) \\ &= \begin{bmatrix} x_1(k_{j'+l}) \\ C + \varepsilon'' \end{bmatrix} \quad \text{for some } \varepsilon'' = \varepsilon''(l) \\ &\leq \Gamma^L(D_2(x(k_{j'}))) \\ &\leq D_1 \circ \Gamma^L \circ D_2(x(k_{j'})), \end{aligned}$$

where $\varepsilon'' < \varepsilon'$. The second component of the last line can be estimated

$$\begin{aligned} [D_1 \circ \Gamma^L \circ D_2(x(k_{j'}))]_2 &\geq (\mathrm{id} + \rho_1)(C + \varepsilon'') \\ &> C + \rho_1(C) > C + \varepsilon > x_2(k_{j'}) \end{aligned}$$

and for the first component we find

$$[D_1 \circ \Gamma^L \circ D_2(x(k_{j'}))]_1 > x_1(k_{j'+l}) > x_1(k_{j'}).$$

Together this gives a point $x(k_{j'}) > 0$ such that

$$D_1 \circ \Gamma^L \circ D_2(x(k_{j'})) \geq x(k_{j'})$$

in contradiction to $D_1 \circ \Gamma^L \circ D_2 \not\geq \mathrm{id}$. Hence we must have $C = 0$. This proves the lemma. ∎

In the following example all assumptions of Lemma 6.1 are satisfied, but nonetheless only the second component of trajectories with certain initial values converge to 0, while the first component diverges.

Example 6.2: Fix some real constants $\lambda \in]0, 1[$ and $\mu \geq 0$. Let $\Gamma : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ be given by

$$\Gamma\left(\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}\right) = \begin{bmatrix} \lambda s_1 + s_1^2 s_2 + \mu s_2 \\ \lambda s_2 \end{bmatrix}$$

for all $s = (s_1, s_2)^T \in\in \mathbb{R}^2_+$.

The map Γ fulfills $\Gamma(0) = 0$, is continuous, monotone, and for $D = (1 + \alpha)\mathrm{id}_{\mathbb{R}^2_+}$, where $0 < \alpha < 1/\lambda - 1$, it satisfies $D \circ \Gamma \not\geq \mathrm{id}$, as can easily be seen. (Just consider the cases ($s_1 = 0$, $s_2 > 0$) and $s_2 = 0$ separately.) If $s^0 = (s_1^0, s_2^0)^T > 0$ is such that $s_1^0 > 1/(\lambda s_2^0)$, then the trajectory of (4) starting in s^0 is unbounded in the first component: The condition $s_1(k) > 1/(\lambda s_2(k))$ with $s(k) > 0$ implies $s_1(k+1) = s_1(k)(\lambda + s_1(k)s_2(k) + \mu s_2(k)) > s_1(k)/\lambda > 1/(\lambda^2 s_2(k)) = 1/(\lambda s_2(k+1))$ and clearly $s(k+1) > 0$. By induction we obtain a trajectory that converges to 0 in the second component and diverges in the first one. Hence the monotone system induced by Γ is not 0-GAS. Geometrically we have

$$\begin{aligned} \Omega_1 &= \{s \in \mathbb{R}^2_+ : \lambda s_1 + s_1^2 s_2 + \mu s_2 < s_1\} \\ &= \left\{ s \in \mathbb{R}^2_+ : s_1 > 0,\ s_2 < \frac{(1-\lambda)s_1}{s_1^2 + \mu} \right\} \\ \Omega_2 &= \{s \in \mathbb{R}^2_+ : \lambda s_2 < s_2\} = \{s \in \mathbb{R}^2_+ : s_2 \neq 0\}. \end{aligned}$$

The picture in Figure 1 is drawn for $\lambda = 1/2$ and $\mu = 1/16$.

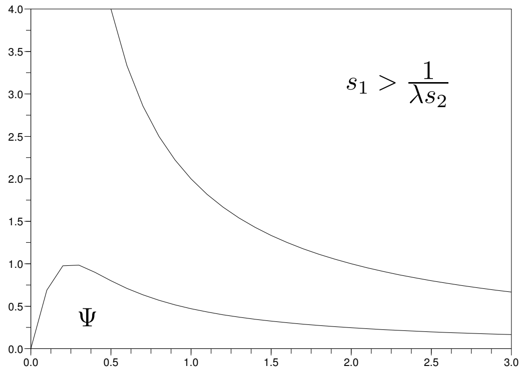

Fig. 1. Attracting and repelling sets in Example 6.2

In the next result the assumptions on LL and UR (acronyms of lower-left and upper-right) roughly state that the set Ψ has eventually to stay away from the coordinate axes.

Proposition 6.3: Let $\Gamma : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ be monotone and continuous, $\Gamma(0) = 0$. Suppose there exists an operator $D : \mathbb{R}^2_+ \to \mathbb{R}^2_+$, $D = \operatorname{diag}_2(\mathrm{id} + \rho)$, $\rho \in \mathcal{K}_\infty$, such that $D \circ \Gamma \not\geq \mathrm{id}$. Suppose that the set Ψ^Γ satisfies the conditions

(i) $LL_2(r) := \inf\{s_2 : (s_1, s_2)^T \in \Psi^\Gamma \cap S_r$ for some $s_1 > 0\}$ is nondecreasing in $r > 0$ and there exists an $R_1 > 0$, such that $LL_2(R_1) > 0$.

(ii) $UR_1(r) := \inf\{s_1 : (s_1, s_2)^T \in \Psi^\Gamma \cap S_r$ for some $s_2 > 0\}$ is nondecreasing in $r > 0$ and there exists an $R_2 > 0$, such that $UR_1(R_2) > 0$.

Then system (4) is 0-GAS.

Proof: By our assumptions, there exists $R = \max\{R_1, R_2\}$ such that $\varepsilon := \min\{LL_2(R), UR_1(R)\} > 0$.

For each initial value x_0 of (4) by Lemma 6.1 we find an index k such that one component of $x(k)$ is less that ε while the other is greater than R. So without loss of generality assume $x_1(k) > R$ and $x_2(k) < \varepsilon$.

Hence we find a point $y \geq x(k)$, such that $y \in \Psi^\Gamma$. By Corollary 5.11 the proof is complete. ■

VII. Exploiting the graph structure of matrices with entries in $(\mathcal{K} \cup \{0\})$

This section is devoted to the special case that Γ is a matrix with entries in $(\mathcal{K} \cup \{0\})$. Of course, if every entry of Γ is bounded, then by Lemma 5.13 it follows that (4) is 0-GAS.

So here we consider $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$, which will lead to the satisfying result that if Γ is irreducible, then Γ is nowhere increasing if and only if system (1) is 0-GAS (Theorem 7.7).

If Γ is reducible we reformulate the problem in terms of a diagonal robustness operator D. There exists a diagonal robustness operator D such that $D \circ \Gamma$ is nowhere increasing if and only if there exists a diagonal robustness operator $\tilde{D}$ such that $\tilde{D} \circ \Gamma$ defines a 0-GAS system (Theorem 7.10).

In later proofs in this section we will rely on the following two facts:

Lemma 7.1: Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$. Let A_{Γ^k} be the adjacency matrix of Γ^k for $k \in \mathbb{N}$. Then

$$(A^k_\Gamma)_{ij} \neq 0 \iff (A_{\Gamma^k})_{ij} \neq 0.$$

Lemma 7.2: Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$. Let $\alpha_i \in \mathcal{K}_\infty$ for $i = 1, \ldots, n$, and define $D = \operatorname{diag}_n(\alpha_1, \ldots, \alpha_n)$. Let $P \in \{0, 1\}^{n \times n}$ be a permutation matrix and let $T = P \cdot D \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$. Then the graphs of Γ, $D \circ \Gamma$, and $\Gamma \circ D$ coincide and the graph of $T^{-1} \circ \Gamma \circ T$ is the graph of the former maps with renumbered vertices.

The proofs are not difficult (just write down the respective maps explicitly) and left to the reader.

A. The irreducible case

Lemma 7.3: Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$ be primitive and $\Gamma \not\geq \mathrm{id}$. Then for any $x \in \mathbb{R}^n_+$ the sequence $\{\Gamma^k(x)\}_{k \in \mathbb{N}}$ is bounded.

Proof: Suppose there exists an $x \in \mathbb{R}^n_+$ such that $\limsup_{k \to \infty} \|\Gamma^k(x)\| = \infty$ and $\Gamma \not\geq \mathrm{id}$. Let $\{e_i\}_{i=1,\ldots,n}$ denote the standard basis of $\mathbb{R}^n$. Since Γ is primitive there is a $k_0 \in \mathbb{N}$ such that the graph $G(\Gamma^{k_0})$ is fully connected (i.e., any $(i,j) \in E(\Gamma^{k_0})$ for $i, j = 1, \ldots, n$), see Lemma 7.1. Hence there exists a $T \in \mathbb{R}$, $T > 0$, such that for all $t \in \mathbb{R}$, $t > T$, and some fixed $i \in \mathbb{N}$ we have $\Gamma^{k_0}(t \cdot e_i) > x$ (recall, this means $P_j(\Gamma^{k_0}(t \cdot e_i)) > x_j$). Since $\limsup\limits_{k \to \infty} \|\Gamma^k(x)\| = \infty$ there exists a $k_1 \in \mathbb{N}$ and index i_0 such that

$$P_{i_0}(\Gamma^{k_1}(x)) > T.$$

By monotonicity of Γ we have

$$\begin{aligned}\Gamma^{k_0} \circ \Gamma^{k_1}(x) &\geq \max_{i \in \mathbb{N}} \Gamma^{k_0}(P_i(\Gamma^{k_1}(x)) \cdot e_i) \\ &\geq \Gamma^{k_0}(P_{i_0}(\Gamma^{k_1}(x)) \cdot e_{i_0}) \geq \Gamma^{k_0}(T \cdot e_{i_0}) \geq x.\end{aligned}$$

This contradicts $\Gamma^k \not\geq \mathrm{id}$ for all $k \geq 1$ which is asserted by Lemma 5.3. The proof is complete. ■

Remark 7.4 (coordinate change): By a change of coordinates, using a permutation matrix $P \in \{0, 1\}^{n \times n}$, it is possible to rearrange the adjacency matrix A of Γ. There are two main cases:

Either A is irreducible, or it is not. In the latter case, A is similar to an upper triangular block matrix of the form

$$PAP^T = \tilde{A} = \begin{bmatrix} A_{11} & \ldots & \ldots & A_{1l} \\ 0 & A_{22} & \ldots & A_{2l} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & A_{ll} \end{bmatrix},$$

where all diagonal blocks $A_{jj}, j = 1, \ldots, l$ are irreducible or zero.

Now let A be irreducible. By well known results (see, e.g., [3, Chapter 2: Theorems 2.20, 2.30, 2.33],[4]) from graph theory, either A is primitive, or there exists an integer $k \geq 1$, such that A^k is similar to a block diagonal matrix with primitive blocks B_j, $j = 1, \ldots, l$, i.e., there exists a permutation matrix P, such that

$$PA^kP^T = \operatorname{diag}_l(B_1, \ldots, B_l). \tag{10}$$

Lemma 7.5: A trajectory defined by (4) is unbounded if and only if for any $k_0 \in \mathbb{N}$ the sequence defined by

$$y(k+1) := \Gamma^{k_0}(y(k)), \ k \in \mathbb{N}, \quad y(0) = x(0), \tag{11}$$

is unbounded.

The easy proof is omitted.

Lemma 7.5 allows us to consider Γ^{k_0} instead of Γ, when we investigate boundedness of orbits of (4). Now we are able to establish a consequence of Lemma 7.3:

Lemma 7.6: Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$ be irreducible and nowhere increasing. Then for any $x \in \mathbb{R}^n_+$ the sequence $\{\Gamma^k(x)\}_{k \in \mathbb{N}}$ is bounded.

Proof: Let A be the adjacency matrix of Γ and assume A^k is similar to the right hand side form of (10) for some $k \in \mathbb{N}$. Let I_j, $j = 1, \ldots, l$, denote the indices corresponding to B_j. Since we have $P_{I_j}(\Gamma^k(x|_{I_j})) \leq P_{I_j}(\Gamma^k(x))$ and each $P_{I_j}(\Gamma^k(\cdot|_{I_j}))$ satisfies the premises of Lemma 7.3, by Lemma 7.5 the problem reduces to l parallel applications of Lemma 7.3. ∎

Theorem 7.7: Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ be irreducible. Then $\Gamma \not\geq \mathrm{id}$ if and only if system (4) is globally asymptotically stable in 0.

Proof: $\Leftarrow$: If Σ is 0-GAS, then in particular each trajectory is attracted to zero, hence Proposition 5.2 establishes $\Gamma \not\geq \mathrm{id}$.

$\Rightarrow$: Zero is an equilibrium point and each trajectory of the system is globally attracted to zero by Lemma 7.6 and Lemma 5.4. Stability follows from Lemma 5.10. ∎

B. The reducible case

Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ be reducible. Without loss of generality, by Remark 7.4, we may assume that the adjacency matrix of $G = G(\Gamma)$ is

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{12}$$

where A_{11} is irreducible, or equivalently stated

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{bmatrix} \tag{13}$$

with Γ_{11} an irreducible $n_1 \times n_1$ matrix with entries of class $(\mathcal{K}_\infty \cup \{0\})$ and Γ_{12} and Γ_{22} some $n_1 \times n_2$ and, respectively, $n_2 \times n_2$ matrices with entries in $(\mathcal{K}_\infty \cup \{0\})$. Note that $\Gamma_{11}(s_1) \not\geq s_1$ and $\Gamma_{22}(s_2) \not\geq s_2$ for all $s_i \in \mathbb{R}^{n_i}_+, s_i \neq 0$. For a fixed $s(0) \in \mathbb{R}^n_+$ we recursively define a sequence $\{s(k)\}_{k\in\mathbb{N}_0}$ by

$$s(k+1) := \Gamma(s(k)), k \in \mathbb{N}_0. \tag{14}$$

We also consider the projected sequences

$$s_i(k) := P_{n_i}(s(k)), \quad \text{for } i = 1, 2. \tag{15}$$

This motivates the following statement.

Lemma 7.8: Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ be reducible and of the form (13). Suppose that there exists a $D = \mathrm{diag}_n(\mathrm{id} + \rho)$, $\rho \in \mathcal{K}_\infty$, such that $D \circ \Gamma \not\geq \mathrm{id}$. Suppose there are $l > 1$ blocks on the diagonal, all irreducible and the i^{th} block of dimension n_i, $i = 1, \ldots, l$, such that we have $n = n_1 + \ldots + n_l$. For $i = 1, \ldots, l-1$, we define the sets $M^i_\Gamma(s_2)$ for $s_2 \in \mathbb{R}^{n_{i+1}+\ldots+n_l}_+$ by

$$M^i_\Gamma(s_2) := M^i_\Gamma(s_2, D) := \Bigg\{ s_1 \in \mathbb{R}^{n_i}_+ : \\ P_{n_i} \circ D \circ \Gamma \circ S_{n_i}(s_1) \geq \\ P_{n_i} \circ \Gamma \circ S_{n_i+\ldots+n_l}\left(\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}\right) \Bigg\}.$$

Then we have

1) for any $s_2 \in \mathbb{R}^{n_{i+1}+\ldots+n_l}_+, s_1, t_1 \in \mathbb{R}^{n_i}_+$ it holds that $s_1 \in M^i_\Gamma(s_2) \implies s_1 + t_1 \in M^i_\Gamma(s_2)$,
2) for any $s_2, t_2 \in \mathbb{R}^{n_{i+1}+\ldots+n_l}_+$ we have $s_2 \geq t_2 \implies M^i_\Gamma(s_2) \subset M^i_\Gamma(t_2)$, and
3) $\mathbb{R}^{n_1}_+ = \bigcup_{s_2\geq 0} M^i_\Gamma(s_2) \supset \bigcup_{s_2>0} M^i_\Gamma(s_2) \supset \mathbb{R}^{n_1}_+ \setminus \{0\}$.

Proof: The proof is straightforward and thus omitted. ∎

Define a subsystem Σ_2 of system (4) as the projected dynamical system on $\mathbb{R}^{n_2}_+$:

$$\Sigma_2 : \ x_2(k+1) = P_{n_2}(x(k+1)) = \Gamma_{22}(x_2(k)), \ k \in \mathbb{N}. \tag{16}$$

Lemma 7.9: Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ be reducible. Let $D = \mathrm{diag}_n(\mathrm{id}+\rho)$ for some $\rho \in \mathcal{K}_\infty$. Assume that $D\circ\Gamma \not\geq \mathrm{id}$. Let Γ satisfy equation (13), such that Γ_{11} is irreducible. If system (4) is such that system (16) is globally attracted to zero, then for system (4) the origin of $\mathbb{R}^n_+$ is GAS.

Proof: First note, that by our assumptions the system $s_{k+1} = (D_1 \circ \Gamma_{11})(s_k)$, $k \geq 0$, with initial value s_0 is 0-GAS, since $D_1 \circ \Gamma_{11} \not\geq \mathrm{id}$ satisfies the premises of Theorem 7.7.

Fix an initial value $x(0) = (x_1(0)^T, x_2(0)^T)^T$. Fix some $a \in \mathbb{R}^{n_1}_+, a > 0$. By 3 of Lemma 7.8 there exists a $t_2 \in \mathbb{R}^{n_2}_+, t_2 > 0$, such that

$$\{x_1(k) : x_1(k) \not\leq a\} \subset M^1_\Gamma(t_2).$$

Further, there exists a $k_1 \in \mathbb{N}$, such that $x_2(k) \leq t_2$ for all $k \geq k_1$.

Hence, for $k \geq k_1$ and $x_1(k) \not\leq a$ we have $x_1(k+1) \leq D_1 \circ \Gamma_{11}(x_1(k))$. Inductively, as long as $x_1(k+p) \not\leq a$, $p = 0, \ldots, l-1$, we get $x_1(k+l) \leq (D_1 \circ \Gamma_{11})^l(x_1(k))$, so at some point we arrive at an $l \in \mathbb{N}$ such that $x_1(k+l) \leq a$ (since $D_1 \circ \Gamma_{11}$ is assumed to by 0-GAS).

Let $U_a = \{x \in \mathbb{R}^{n_1}_+ : x \leq a\}$, $W = \{z \in \mathbb{R}^{n_1}_+ : z \leq P_{n_1} \circ \Gamma((a^T, t_2^T)^T)\}$.

Since $D_1 \circ \Gamma_{11}$ is 0-GAS, there exists a bounded set $W_2 \supset W$: $D_1 \circ \Gamma_{11}(W_2) \subset W_2$. Without loss of generality W_2 is such that $x \in W_2$ implies $y \in W_2$ for all $0 \leq y \leq x$, since $y \leq x \in \Gamma(W_2)$ implies $\Gamma(y) \leq \Gamma(x) \in \Gamma(W_2) \subset W_2$.

Now if $x_1(k) \in U_a$, then $x_1(k+1) \in W \subset W_2$. There remain two cases:

1) $x_1(k+1) \in U_a$. Then $x_1(k+2) \in W \subset W_2$.
2) $x_1(k+1) \notin U_a$. Hence $x_1(k+1) \in W \subset W_2$ and therefore we find $x_1(k+1) \in M^1_\Gamma(t_2)$. Hence $x_1(k+2) \leq D_1 \circ \Gamma_{11}(x_1(k+1)) \in W_2$.

It follows, that $\{x_1(k)\}_{k\in\mathbb{N}}$ is bounded. Now we apply Lemma 5.4 which gives us $x_1(k) \xrightarrow{k\to\infty} 0$. Since we already know that $x_2(k) \xrightarrow{k\to\infty} 0$, we can deduce stability from Lemma 5.10. This completes the proof. ∎

The main result of this section is the following:

Theorem 7.10: Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$. Then the following are equivalent:

1) There exists a $\rho \in \mathcal{K}_\infty$ such that for $D = \mathrm{diag}_n(\mathrm{id}+\rho)$ we have $D \circ \Gamma \not\geq \mathrm{id}$.
2) There exists a $\rho \in \mathcal{K}_\infty$ such that for $D = \mathrm{diag}_n(\mathrm{id}+\rho)$ the discrete dynamical system defined by

$$x(0) \in \mathbb{R}^n_+, \ x(k+1) := D\circ\Gamma(x(k)), \ k \in \mathbb{N}_0, \tag{17}$$

is globally asymptotically stable in 0.

Proof: 1⇒2: First note, that by Lemma 4.2 we can decompose $D = \tilde{D} \circ \hat{D}$. We then apply Lemma 7.9 inductively.

2⇒1: This follows by an application of Proposition 5.2 with Γ replaced by $D \circ \Gamma$. ∎

In [4] an example was presented, that $\Gamma \not\geq \mathrm{id}$ alone does not imply that system (1) is 0-GAS.

VIII. Applications to large-scale interconnections of input-to-state stable systems

In [4] the authors showed that existence of a $\rho \in \mathcal{K}_\infty$, such that for $D = \mathrm{diag}_n(\mathrm{id} + \rho)$, the condition

$$D \circ \Gamma \not\geq \mathrm{id} \tag{18}$$

implies input-to-state stability (ISS) of a large scale N-dimensional dynamical system. This large-scale system is given by

$$\Sigma : \quad \dot{x} = f(x, u), \quad x \in \mathbb{R}^N, u \in \mathbb{R}^M, \tag{19}$$

and can be decomposed into n smaller N_i-dimensional systems Σ_i, $i = 1, \ldots, n$, with $N = N_1 + \ldots + N_n$, $M = M_1 + \ldots + M_n$, given by

$$\Sigma_i : \quad \begin{aligned} &\dot{x}_i = f(x_1, \ldots, x_n, u_i), \\ &x_j \in \mathbb{R}^{N_j}, \; j = 1, \ldots, n, \quad u_i \in \mathbb{R}^{M_i}, \end{aligned} \tag{20}$$

where $i = 1, \ldots, n$. Suppose each system Σ_i satisfies the standard assumptions for existence and uniqueness of solutions and is forward-complete.

Assume that each Σ_i fulfills the ISS condition: There exists a function $\beta_i \in \mathcal{KL}$, and functions $\gamma_{ij}, \alpha_i \in (\mathcal{K}_\infty \cup \{0\})$, with $\gamma_{ii} \equiv 0$, such that for all $x_i(0) \in \mathbb{R}^{N_i}$ and $t \in \mathbb{R}_+$ the estimate

$$\begin{aligned} |x_i(t)| \leq \beta_i(|x_i(0)|, t) + \sum_{i=1}^{n} \gamma_{ij}(\|x_j\|_{L_\infty[0,t]}) \\ + \alpha_i(\|u_i\|_{L_\infty[0,t]}) \end{aligned} \tag{21}$$

holds. Here Γ is the *nonlinear gain matrix*, simply defined by $\Gamma = (\gamma_{ij})$. The functions γ_{ij} and α_i are called gains in this context, hence the name.

Now by putting the results of the previous sections together, we can state the main result of this section:

Theorem 8.1: Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$. Then the following are equivalent:

1) There exists a $\rho \in \mathcal{K}_\infty$ such that for $D = \mathrm{diag}_n(\mathrm{id} + \rho)$ we have $D \circ \Gamma \not\geq \mathrm{id}$.
2) There exists a $\rho \in \mathcal{K}_\infty$ such that for $D = \mathrm{diag}_n(\mathrm{id} + \rho)$ the discrete dynamical system defined by

$$x(0) \in \mathbb{R}^n_+, \; x(k+1) := D \circ \Gamma(x(k)), \; k \in \mathbb{N}_0, \tag{22}$$

is globally asymptotically stable in 0.

Both imply ISS of the corresponding large-scale dynamical system Σ defined by (19).

Proof: The equivalence has already been proved in Theorem 7.10. In [4] the authors proved that 1) implies that Σ is ISS. ∎

This clarifies and establishes the role of the discrete dynamical system associated with Γ as a sufficient stability criterion for the large-scale system Σ.

A. The equivalence of 0-GAS and $\mathbf{D} \circ \mathbf{\Gamma} \not\geq$ id for more general monotone maps

In this section the graph structure associated to matrices with entries in $(\mathcal{K}_\infty \cup \{0\})$ played an important role. It is possible to define such a graph also for more general maps. For example, Angeli and Sontag define the signed incidence graph of a monotone map in [2]. For matrices with entries in $(\mathcal{K}_\infty \cup \{0\})$ this definition agrees with our definition of Section III.

A natural question to ask is, if the equivalence stated in Theorem 7.10 does hold for more general monotone maps Γ, which also possess an embedded graph structure, like the incidence graph of Angeli and Sontag.

The following example shows, that there are monotone maps possessing an incidence graph, such that at least assertion 1 of Lemma 7.8 does not hold:

Example 8.2: Let $\Gamma : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ be given by

$$\Gamma(\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}) = \begin{bmatrix} (\lambda + s_2)s_1 + \mu s_2 \\ \lambda s_2 \end{bmatrix}, \lambda \in]0, 1[, \mu > 0.$$

The set of edges of the incidence graph is $E = \{(1,1), (2,1), (2,2)\}$. For $0 < \alpha < 1/\lambda - 1$ and $D = (1 + \alpha)\mathrm{id}_{\mathbb{R}^2_+}$ we clearly have $D \circ \Gamma \not\geq \mathrm{id}$. Now look at the set $M(s_2)$ as defined in Lemma 7.8 for $s_2 = 1 - \lambda$. We find that $\Gamma_1(s) = s_1 + \mu(1 - \lambda)$ is less or equal to $(D \circ \Gamma)_1 \circ (s_1, 0)^T = (1 + \alpha)\lambda s_1$ if and only if

$$s_1 \leq \frac{\mu(1 - \lambda)}{1 - (1 + \alpha)\lambda}. \tag{23}$$

Thus $s_1 \in M(1 - \lambda)$ if and only if (23) holds. This clearly violates property 1 of Lemma 7.8, which is an important ingredient in the proof in Lemma 7.9.

IX. Conclusions

For monotone maps Γ on the positive orthant of $\mathbb{R}^n$ it has been shown that if the induced discrete dynamical system is 0-GAS, the inequality $\Gamma \not\geq \mathrm{id}$ must hold. For the converse implication we have to make stronger assumptions on Γ. For matrices with entries in $(\mathcal{K}_\infty \cup \{0\})$ an equivalence relation was obtained.

Some questions remain open. It is unclear if for $n > 2$ there are n-dimensional extensions of Lemma 6.1. Also, is it possible to restate the condition on Γ in Proposition 6.3 in a way that is easier to check?

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