A LYAPUNOV ISS SMALL-GAIN THEOREM FOR STRONGLY CONNECTED NETWORKS

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Abstract: We consider strongly connected networks of input-to-state stable (ISS) systems. Provided a small gain condition holds it is shown how to construct an ISS Lyapunov function using ISS Lyapunov functions of the subsystems. The construction relies on two steps: The construction of a strictly increasing path in a region defined on the positive orthant in $\mathbb{R}^n$ by the gain matrix and the combination of the given ISS Lyapunov functions of the subsystems to a ISS Lyapunov function for the composite system.
Novelties are the explicit path construction and that all the involved Lyapunov functions are nonsmooth, i.e., they are only required to be locally Lipschitz continuous. The existence of a nonsmooth ISS Lyapunov function is qualitatively equivalent to ISS.

Keywords: Lyapunov function, nonlinear control systems, nonlinear gain, stability criteria, decentralized systems.

1. INTRODUCTION

In this paper we are interested in the stability of a network of nonlinear input to state stable (ISS) systems. A nonlinear small gain theorem for networks of input-to-state stable (ISS) systems was obtained in Dashkovskiy et al. (2007). Here we provide a constructive method to find a nonsmooth ISS Lyapunov function for a composite system, when the ISS Lyapunov functions and nonlinear gains for the subsystems are all known. This result is particularly useful, since the knowledge of a Lyapunov function directly leads to knowledge of invariant sets and allows for different controller design methods, see, e.g., Khalil (1996). A main step of the construction was already carried out in Dashkovskiy et al. (2006c). Namely, it was shown how to construct a nonsmooth ISS Lyapunov function, if a strictly increasing function $\sigma : \mathbb{R}_+ \to \mathbb{R}^n_+$ exists such that $D(\Gamma(\sigma(t))) < \sigma(t)$ for all $t > 0$. Here Γ is the gain matrix, and D is a diagonal scaling operator. In Dashkovskiy et al. (2006c) the existence of such a function was shown only for the case of three interconnected systems. The case of two systems in a feedback loop was considered in Jiang et al. (1994) and the construction of Lyapunov functions for this case was presented in Jiang et al. (1996).

The small gain condition derived in Dashkovskiy et al. (2007) leads to interesting invariance properties of the map defined by Γ, which allow a construction of the desired σ. Here we are going to construct a σ, that is differentiable almost everywhere. The overall Lyapunov function is then obtained as a weighted maximum of the ISS Lyapunov functions of the subsystems similar to Jiang et al. (1996). As a consequence the constructed Lyapunov function is not differentiable, so that we resort to nonsmooth formulations of ISS Lyapunov functions. An alternative would be to use a smooth approximation, which is possible in principle. We avoid this as it does not add to the understanding of our construction.

In Proposition 12 we construct a piecewise linear and strictly increasing function $\sigma_s : [0,1] \to \mathbb{R}^n_+$ up to some predetermined radius, provided that Γ is irreducible. If Γ is even primitive, then this function can be extended to a function $\sigma \in \mathcal{K}^n_\infty$. If Γ is only irreducible, this function σ can still be defined, but under slightly stronger assumptions, see Theorem 14.

2. NOTATION

Let $\mathcal{K} = \{f : \mathbb{R}_+ \to \mathbb{R}_+ : f \text{ is continuous, strictly increasing and } f(0) = 0\}$ and $\mathcal{K}_\infty = \{f \in \mathcal{K} : f \text{ is unbounded}\}$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$, if it is of class $\mathcal{K}$ in the first component and strictly decreasing to zero in the second component.

A matrix $\Gamma = (\gamma_{ij}) \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ defines a map on $\mathbb{R}^n_+$ via $\Gamma(s)_i = \sum_{j=1}^n \gamma_{ij}(s_j)$, for $s \in \mathbb{R}^n_+$, in analogy to matrix vector multiplication.

The adjacency matrix $A_\Gamma = (a_{ij})$ of a matrix $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ is defined by $a_{ij} = 0$ if $\gamma_{ij} \equiv 0$ and $a_{ij} = 1$ otherwise. We say that the matrix Γ is *primitive, irreducible* or *reducible* if and only if A_Γ is primitive, irreducible or reducible. See e.g. Berman and Plemmons (1979) for definitions.

On $\mathbb{R}^n_+$ we use the partial order induced by the positive orthant. For vectors $x, y \in \mathbb{R}^n_+$ we define

$$x \geq y :\iff x_i \geq y_i \text{ for } i = 1,\dots,n,$$
$$x > y :\iff x_i > y_i \text{ for } i = 1,\dots,n, \text{ and}$$
$$x \gneqq y :\iff x \geq y \text{ and } x \neq y.$$

A map $\Delta : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is *monotone* if $x \leq y$ implies $\Delta(x) \leq \Delta(y)$. Clearly $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ induces a monotone map. For $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $\Delta : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ we write $\Gamma \geq \Delta$ if for all $x \in \mathbb{R}^n_+$ we have $\Gamma(x) \geq \Delta(x)$. Similarly, we write $\Gamma \not\geq \Delta, \Gamma > \Delta$, respectively $\Gamma \gneqq \Delta$, if for all $x \in \mathbb{R}^n_+ \setminus \{0\}$ we have $\Gamma(x) \not\geq \Delta(x)$, $\Gamma(x) > \Delta(x)$, respectively $\Gamma(x) \gneqq \Delta(x)$. Here $x \not\geq y$ means that for at least one component i the inequality $x_i < y_i$ holds.

For monotone maps Γ on $\mathbb{R}^n_+$ we define the following sets:

$$\Omega(\Gamma) = \{x \in \mathbb{R}^n_+ : \Gamma(x) < x\},$$
$$\Omega_i(\Gamma) = \{x \in \mathbb{R}^n_+ : \Gamma(x)_i < x_i\},$$
$$\Psi(\Gamma) = \{x \in \mathbb{R}^n_+ : \Gamma(x) \leq x\}.$$

If no confusion arises we will omit the reference to Γ. Note that for general monotone maps we have $\overline{\Omega} \subsetneq \Psi$, but for $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ we have equality.

By $|\cdot|$ we denote the 1-norm on $\mathbb{R}^n$ and by S_r the induced sphere of radius r in $\mathbb{R}^n$ intersected with $\mathbb{R}^n_+$, which is an n-simplex. By $U_\varepsilon(x)$ we denote the open neighborhood of radius ε around x with respect to the Euclidean norm $\|\cdot\|$.

For our construction we will need the notions of proximal subgradient and nonsmooth ISS Lyapunov functions, c.f. Clarke et al. (1998), Clarke (2001). Also we need some results from nonsmooth analysis.

Definition 1. A vector $\zeta \in \mathbb{R}^N$ is a proximal subgradient of a function $\phi : \mathbb{R}^N \to (-\infty, \infty]$ at $x \in \mathbb{R}^N$ if there exists a neighborhood $U(x)$ of x and a number $\sigma \geq 0$ such that

$$\phi(y) \geq \phi(x) + \langle \zeta, y - x\rangle - \sigma|y - x|^2 \quad \forall y \in U(x).$$

The set of all proximal sub-gradients at x is the proximal sub-differential of ϕ at x and is denoted by $\partial_P \phi(x)$.

3. INPUT-TO-STATE STABILITY

We consider a finite set of interconnected systems

$$\Sigma_i : \dot{x}_i = f(x_1, \dots, x_n, u), \; f_i : \mathbb{R}^{N+M} \to \mathbb{R}^{N_i}, \tag{1}$$

$i = 1,\dots,n$, where $x_i \in \mathbb{R}^{N_i}$, $u \in \mathbb{R}^M$, $\sum N_i = N$.

If we consider one of the systems, indexed by i, and interpret the variables x_j, $j \neq i$, and u as unrestricted inputs, then this system is assumed to have unique solutions defined on $[0,\infty)$ for all L^∞-inputs $x_j : [0,\infty) \to \mathbb{R}^{N_j}, j \neq i$, and $u : [0,\infty) \to \mathbb{R}^M$.

We write the interconnection of systems (1) as

$$\Sigma : \; \dot{x} = f(x,u), \quad f : \mathbb{R}^{N+M} \to \mathbb{R}^N, \tag{2}$$

where $x = (x_1^T, \dots, x_n^T)^T$.

We will impose ISS conditions on the subsystems given by (1) and we are interested in conditions guaranteeing ISS of the interconnected system (2). To this end we will construct an ISS Lyapunov function for (2).

Definition 2. (ISS Lyapunov function). A smooth function $V : \mathbb{R}^N \to \mathbb{R}_+$ is an *ISS Lyapunov function* of (2) if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}_\infty$, and a positive definite function α such that

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^N, \tag{3}$$

$$V(x) \geq \chi(|u|) \Longrightarrow \nabla V(x) \cdot f(x,u) \leq -\alpha(V(x)). \tag{4}$$

The function χ is called a *Lyapunov-gain.* System (2) is *input-to-state stable* (ISS) if it has an ISS Lyapunov function.

It is well known, see Sontag and Wang (1996), that the existence of an ISS Lyapunov function is equivalent to the system being ISS in the following sense:

There exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all initial conditions $x_0 \in \mathbb{R}^N$ and all L_∞-inputs $u(\cdot)$ it holds that

$$|x(t)| \le \beta(|x_0|, t) + \gamma(\|u\|_\infty), \quad \text{for all } t \ge 0. \quad (5)$$

In our context we need a notion of ISS Lyapunov functions that avoids the assumption of differentiability. This is just a reformulation of (4).

Definition 3. A continuous function $V : \mathbb{R}^N \to \mathbb{R}_+$ is a *nonsmooth ISS Lyapunov function* of system (2) if there exist functions ψ_1, ψ_2 of class $\mathcal{K}_\infty$ such that V satisfies (3) and such that there exists a positive-definite function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ and a class $\mathcal{K}_\infty$-function χ such that

$$\sup_{u:\, V(x) \ge \chi(|u|)} \langle f(x,u), \zeta \rangle \le -\alpha(V(x)), \quad (6)$$

for all $\zeta \in \partial_P V(x)$, and all $x \neq 0$.

Here $\partial_P V(x)$ denotes the proximal subgradient of V at x. See also (Clarke et al., 1998, p. 188 and Theorem 4.6.3). Note that if V is locally Lipschitz continuous then almost everywhere in $\mathbb{R}^N$ we have (4) if (6) holds. Conversely and still assuming that V is locally Lipschitz, if (4) holds at every point of differentiability of V, then (6) is true.

Clearly ISS in the sense of (5) implies the existence of a nonsmooth ISS Lyapunov function. The converse is also true:

Theorem 4. Let V be a nonsmooth ISS Lyapunov function as in Definition 3 for System (2). Then there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that (5) holds, i.e., system (2) is ISS in the "original sense".

The proof works as the in the known smooth case and is omitted for reasons of space.

In analogy to Definition 2 we extend the ISS notion to the subsystems: We say that the subsystems defined by (1) are ISS, if for $i = 1, \dots, n$ there exist smooth ISS Lyapunov functions $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ and functions $\psi_{1i}, \psi_{2i} \in \mathcal{K}_\infty$, $\chi_{ij} \in (\mathcal{K}_\infty \cup \{0\})$, and $\chi_i \in \mathcal{K}_\infty$, and positive definite functions α_i such that

$$\psi_{1i}(|x_i|) \le V_i(x_i) \le \psi_{2i}(|x_i|), \quad \forall x_i \in \mathbb{R}^{N_i}, \quad (7)$$

and all $x_i \in \mathbb{R}^{N_i}$

$$\begin{aligned} & V_i(x_i) \ge \sum_{j \neq i} \chi_{ij}(V_j(x_j)) + \chi_i(|u|) \\ & \Longrightarrow \nabla V_i(x_i) \cdot f_i(x,u) \le -\alpha_i(V_i(x_i)). \end{aligned} \quad (8)$$

The functions χ_{ij} are called *ISS Lyapunov gains* or simply *gains*, if no confusion arises.

We refer to the subsystems (1) with their respective ISS Lyapunov functions satisfying (7) and (8) as a network of ISS systems. The questions is, whether the composite system (2) is ISS from u to x.

Consider the network of ISS systems given by (1). Setting $\chi_{ii} \equiv 0$, $i = 1, \dots, n$, the gain functions χ_{ij} give rise to an $n \times n$-gain matrix

$$\Gamma := (\chi_{ij}) \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}.$$

Associated to such a network is a graph, whose vertices are the systems and its directed edges (i,j) correspond to inputs going from system j to system i. We will call the network strongly connected if its graph is.

4. MAIN RESULTS

We construct an ISS Lyapunov function under the assumption, that the network is strongly connected, or equivalently, that Γ is irreducible.

Theorem 5. (Lyapunov-type small gain theorem). Consider a strongly connected ISS network as in (1), (7), and (8). Assume there exists a class $\mathcal{K}_\infty$-function η such that for $D = \mathrm{diag}_n(\mathrm{id} + \eta)$ we have

$$D \circ \Gamma(s) \not\ge s, \quad \forall s \in \mathbb{R}^n_+, s \neq 0. \quad (9)$$

Then there exists an ISS Lyapunov function for system (2).

Remark 6. For the case of two systems in a feedback loop condition (9) is equivalent to the small gain condition derived in Jiang et al. (1994), see also Dashkovskiy et al. (2007).

The proof relies on two steps. First we construct a piecewise linear $\sigma \in \mathcal{K}^n_\infty$ with trace in $\Omega(D \circ \Gamma) \cup \{0\}$ for a suitable diagonal operator $D = \mathrm{diag}_n(\mathrm{id} + \alpha)$, $\alpha \in \mathcal{K}_\infty$. Namely, such that for $i = 1, \dots, n$

$$\sigma_i(t) > (\mathrm{id} + \alpha)\Big(\sum_{j=1}^n \chi_{ij}(\sigma_j(t))\Big), \quad \forall t > 0. \quad (10)$$

The second step was done in (Dashkovskiy et al., 2006c, Theorem 6), which we quote as

Proposition 7. Consider an ISS network as in (1), (7), and (8). For each subsystem Σ_i, $i = 1, \dots, n$, let V_i be an ISS Lyapunov function satisfying (7) and (8). Assume there exists a diagonal operator $D = \mathrm{diag}_n(\mathrm{id} + \alpha)$, $\alpha \in \mathcal{K}_\infty$, and a locally Lipschitz path in $\mathbb{R}^n_+$ parameterized by $\sigma \in \mathcal{K}^n_\infty$, satisfying $\sigma(t) \in \Omega(D \circ \Gamma)$ for all $t > 0$ and $(\sigma_i^{-1})'(t) > 0$ for almost all $t > 0, i = 1, \dots, n$. Then the composite system (2) is ISS with ISS Lyapunov function

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\}. \quad (11)$$

Note that by definition the Lyapunov function defined in (11) is not smooth, so that we are constructing an ISS Lyapunov function in the sense of Definition 3. Before proving Theorem 5 we develop some theory for matrices in $(\mathcal{K}_\infty \cup \{0\})^{n\times n}$.

Lemma 8. Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ be such that Γ has no zero rows. Then $0 < r < s \in \mathbb{R}^n_+$ implies $\Gamma(r) < \Gamma(s)$.
If Γ is primitive, then there is a k only depending on Γ such that $s \lneq t$ already implies $\Gamma^k(s) < \Gamma^k(t)$.

Proof. Just compare $\Gamma(r)_i$ with $\Gamma(s)_i$. These are $\sum_{j=1}^n \gamma_{ij}(r_j)$ and $\sum_{j=1}^n \gamma_{ij}(s_j)$, respectively. Since Γ has no zero rows, both sums are non vanishing, and from $r_j < s_j$, for $j = 1, \ldots, n$, we deduce that the first sum is strictly less than the second.
For the second assertion we consider the adjacency matrix $A_\Gamma = (a_{ij})$ of Γ. Since A_Γ is primitive, there exists a $k > 0$ such that $A_\Gamma^k > 0$. It is easy to check, that this is equivalent to $t \mapsto (\Gamma^k(t \cdot e_j))_i \in \mathcal{K}_\infty$ for all $i, j = 1, \ldots, n$. This proves the lemma. □

Now we state some useful properties of the sets Ψ and Ω.

Lemma 9. Assume $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ is such that $\Gamma \ngeq \mathrm{id}$. Then
(i) for all $r > 0$ we have $\Omega \cap S_r \neq \emptyset$.
(ii) If Γ has no zero rows, then $\Gamma^{k+1}(\Omega) \subset \Gamma^k(\Omega) \subset \Omega$ for all $k \geq 0$.
(iii) $\Gamma^{k+1}(\Psi) \subset \Gamma^k(\Psi) \subset \Psi$ for all $k \geq 0$. All these sets are closed.
(iv) For all $k \geq 0$ the sets $\Gamma^k(\Omega) \cup \{0\}$, and $\Gamma^k(\Psi)$ are pathwise connected.
(v) If Γ is irreducible, the set $\Psi_\infty := \Psi_\infty(\Gamma) := \bigcap_{k=0}^\infty \Gamma^k(\Psi)$ is non-empty, and satisfies $\Psi_\infty \cap S_r \neq \emptyset$ for all $r > 0$.
(vi) If Γ is primitive, then there exists a $k > 0$ such that $(\Gamma^k(\Psi) \setminus \{0\}) \subset \Omega$.
(vii) If Γ is irreducible and there exists a $\mathcal{K}_\infty$-function α, such that for $D = \mathrm{diag}_n(\mathrm{id} + \alpha)$ we have $\Gamma \circ D \ngeq \mathrm{id}$, then $\Gamma(\Psi(\Gamma \circ D)) \setminus \{0\} \subset \Omega(\Gamma)$.

A qualitative picture of (i) can be seen in Figure 1.

For the proof of this lemma we need the following:

Theorem 10. (Knaster,Kuratowski,Mazurkiewicz). Let Δ_n denote the unit n-simplex, and for a face σ of Δ_n let $\sigma^{(0)}$ denote the set of vertices of σ.

If a family $\{A_i | i \in \Delta_n^{(0)}\}$ of subsets of Δ_n is such that all the sets are closed or all are open, and each face σ of Δ_n is contained in the corresponding union $\bigcup\{A_i | i \in \sigma^{(0)}\}$, then $\cap_i A_i \neq \emptyset$.

Remark 11. The original proof for closed sets was given in Knaster et al. (1929), while the formu-

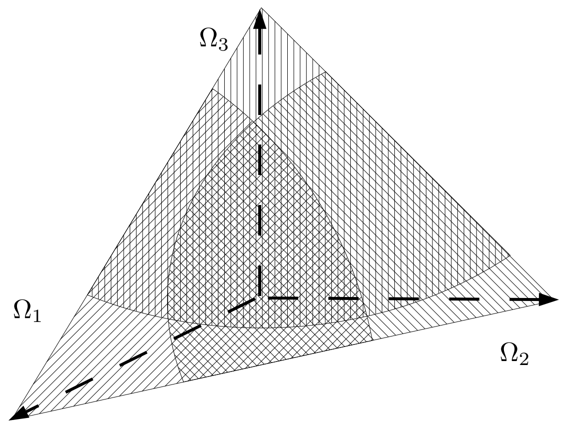

Fig. 1. The set $\Omega_1 \cap \Omega_2 \cap \Omega_3 \cap S_r$ in $\mathbb{R}^3_+$.

lation above is taken from Horvath and Lassonde (1997) and was proved in Lassonde (1990).

Proof of Lemma 9. Some of this can also be found in Dashkovskiy et al. (2006b).

(i) Note that S_r for $r > 0$ is a simplex with vertices $r \cdot e_i$, $i = 1, \ldots, n$. Each (nonempty) face spanned by $r \cdot e_i$, $i \in I \subset \{1, \ldots, n\}$, fulfills the assumptions of the Knaster-Kuratowski-Mazurkiewicz theorem, see Dashkovskiy et al. (2007). I.e., it is contained in the union $\bigcup_I (\Omega_i \cap S_r)$. Then the KKM-theorem implies that $\bigcap_1^n (\Omega_i \cap S_r) \neq \emptyset$.

(ii) Let $s \in \Gamma(\Omega)$, i.e., $s = \Gamma(r)$ for some $r \in \Omega$, that is, $\Gamma(r) < r$. If Γ has no zero rows, then this implies $\Gamma(s) = \Gamma^2(r) < \Gamma(r) = s$, i.e., $s \in \Omega$. The claim follows by induction.

(iii) This may be shown as in (ii).

(iv) Let $s \in \Omega$, then $\lim_{k\to\infty} \Gamma^k(s) = 0$ because $\Gamma^{k+1}(s) < \Gamma^k(s) < \ldots < s$, is a monotone sequence. Its limit point s^* is a fixed point for Γ, hence $s^* = 0$.
Now consider $\lambda \in]0, 1[$ and let $z = (1-\lambda)\Gamma(s) + \lambda s$. Clearly $\Gamma(s) < z < s$. Now apply Γ to obtain $\Gamma^2(s) < \Gamma(z) < \Gamma(s) < z < s$. Hence $z \in \Omega$ and by varying λ from 0 to 1 we get a line segment from $\Gamma(s)$ to s. This interpolation may be performed for all pairs $\Gamma^{k+1}(s) < \Gamma^k(s)$. In this way we construct a piecewise linear strictly increasing path from s to 0. As $s \in \Omega$ was arbitrary this shows the assertion. The argument for Ψ is of course exactly the same.

To complete the proof, for each $r = \Gamma^k(s) \in \Gamma^k(\Omega)$ we may choose a path σ in Ω from s to 0. By (ii) it follows that $\Gamma^k(\sigma)$ is a continuous path from r to 0. This shows the assertion and the same argument applies to $\Gamma^k(\Psi)$, of course.

(v) Since Ψ is nonempty and closed, so are all $\Gamma^k(\Psi)$ by continuity of Γ. Furthermore, the sets $\Gamma^k(\Psi)$ are unbounded. As each $s \in \Gamma^k(\Psi)$ may be connected by a continuous path to 0 it follows that $S_r \cap \Gamma^k(\Psi) \neq \emptyset$ for all $r > 0, k \geq 0$.

As $\Gamma^{k+1}(\Psi) \subset \Gamma^k(\Psi)$ for all $k \geq 0$ using a standard compactness argument we have $\bigcap_{k\geq 0} \Gamma^k(\Psi) \cap S_r \neq \emptyset$ for all $r > 0$.

(vi) First check, that in full analogy to adjacency matrices A, where there exists a $k > 0$ such that the entry $a_{ij}^{(k)} > 0$ of A^k is positive for every $i, j = 1, \ldots, n$, there exists a $k > 0$, such that $t \mapsto \Gamma^k(t \cdot e_j)_i$ is of class $\mathcal{K}_\infty$ for all $i, j = 1, \ldots, n$. Hence $\Gamma(s) \lneqq s$ (and hence $s \neq 0$) implies $\Gamma^{k+1}(s) < \Gamma^k(s)$, because the strict inequality in one component is propagated to every other component.

(vii) This may be seen as in (vi). □

This lemma will be an essential ingredient for the strict monotonicity of the path σ that we want to construct.

An intermediate result is the following, that already implies a semi-global version of Theorem 5, where semi-global means "on arbitrarily large compact sets around the origin".

Proposition 12. Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$, $\Gamma \not\geq \mathrm{id}$, be such that Γ has no zero rows. For every $s \in \Omega$ there exists a continuous and strictly increasing vector function $\sigma_s : [0, 1] \to (\Omega \cup \{0\}) \cap \overline{B_1(0, |s|)}$ with $\sigma_s(0) = 0$ and $\sigma_s(1) = s$. Moreover, each component function is piecewise linear on every interval of the form $[\varepsilon, 1]$, $\varepsilon > 0$.

Figure 2 shows what this looks like in two dimensional space.

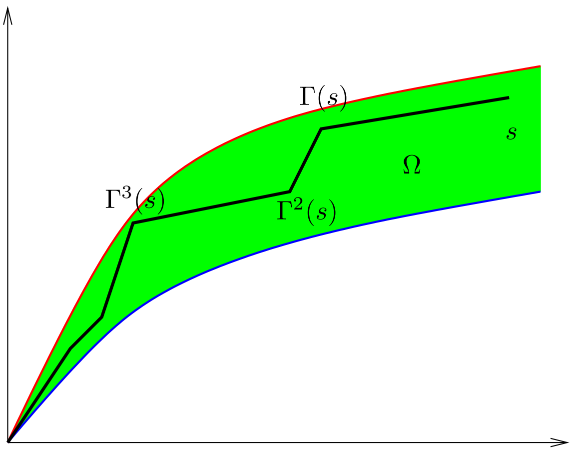

Fig. 2. The path σ_s in $\Omega(\Gamma)$ in $\mathbb{R}_+^2$ can be chosen to be piecewise linear.

Proof. This construction was performed in the proof of Lemma 9 (iv), see Dashkovskiy et al. (2006a) for further details. □

This gives one direction of the path, the other direction is given next.

Theorem 13. Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$, $\Gamma \not\geq \mathrm{id}$, be primitive. Then there exists a piecewise linear and strictly increasing vector function $\sigma : \mathbb{R}_+ \to \Omega \cup \{0\}$ with $\sigma(0) = 0$ and $\lim_{t\to\infty} \sigma(t) = \infty$, i.e., the component functions are of class $\mathcal{K}_\infty$.

Proof. By Lemma 9 (vi) we have $\Psi_\infty \subset \Omega \cup \{0\}$.

Combining the results of Proposition 12 and Lemma 9 we start with $\sigma_s : [0, 1] \to \Psi_\infty$, where $\sigma_s(1) = s \in \Psi_\infty$ and σ_s is piecewise linear.

Since we may always pick a preimage in Ψ_∞ we extend σ_s to a function σ on $\mathbb{R}_+$ by defining $\sigma|_{[0,1]} = \sigma_s$ and

$$\sigma|_{]1,\infty[}(t) = (1-t+\lfloor t\rfloor)\Gamma^{1-\lfloor t\rfloor}(s)+(t-\lfloor t\rfloor)\Gamma^{-\lfloor t\rfloor}(s).$$

It remains to prove unboundedness of the component functions. Assume σ is bounded. Since σ is non decreasing, there must exist a limit point

$$s^* := \lim_{k\to\infty} \sigma(k) = \lim_{k\to\infty} \Gamma(\sigma(k)) = \Gamma(s^*),$$

but since $\sigma(1) > 0$ and σ is non decreasing, and hence $s^* > 0$, this contradicts $\Gamma \not\geq \mathrm{id}$.

So there exists at least one unbounded component of σ, without loss of generality this is the first one. From irreducibility (primitive matrices are also irreducible) we deduce that there exists another unbounded component and inductively we obtain that all components are unbounded.

It follows that the vector function σ constructed above fulfills $\sigma(t) \in \Omega$ for all $t > 0$ and by the same argument as in the proof of Proposition 12 the component functions of σ are strictly increasing and hence of class $\mathcal{K}_\infty$. □

This theorem provides a $\mathcal{K}_\infty^n$-function σ that satisfies

$$\Gamma(\sigma(t)) < \sigma(t), \quad \text{for all } t > 0,$$

for the case that Γ is primitive. Of course, primitivity is quite a restrictive assumption for the topology of the network, that we look at, not every strongly connected network satisfies this assumption. Thus we need to extend the statement to irreducible Γ.

Now the aim is to extend this result to just strongly connected networks. So we have to find such a function σ for irreducible Γ. Recall, that in Theorem 5 we are also given a diagonal operator D and the stronger assertion $\Gamma \circ D \not\geq \mathrm{id}$ instead of $\Gamma \not\geq \mathrm{id}$. (It is easy to see that $D \circ \Gamma \not\geq \mathrm{id}$ and $\Gamma \circ D \not\geq \mathrm{id}$ are equivalent, Dashkovskiy et al. (2006b)). This will come in handy in the next statement.

Theorem 14. Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ be irreducible and assume there exists a function $\alpha \in \mathcal{K}_\infty$, such that for $D = \mathrm{diag}_n(\mathrm{id} + \alpha)$ we have $\Gamma \circ$

$D \not\geq \text{id}$. Then there exists a locally Lipschitz and strictly increasing vector function $\sigma : \mathbb{R}_+ \to \Omega(\Gamma)$ with $\sigma(0) = 0$ and $\lim_{t\to\infty} \sigma(t) = \infty$, i.e., the component functions are of class $\mathcal{K}_\infty$.

Proof. First note that $\Psi_\infty^{\Gamma\circ D} := \bigcap_{k\geq 0}(\Gamma\circ D)^k(\Psi(\Gamma\circ D)) \subset \Omega(\Gamma)$ because of $\Psi(\Gamma \circ D) \subset \Omega(\Gamma)$. The set $\Psi_\infty^{\Gamma\circ D}$ has all the nice properties as Ψ_∞ in Lemma 9. Hence for $s \in \Psi_\infty^{\Gamma\circ D} \subset \Omega(\Gamma)$ there exists an ascending sequence $\{z_k\}_{k\geq 0} \subset \Psi_\infty^{\Gamma\circ D}$, satisfying

$$z_0 = s \quad \text{and} \tag{12}$$
$$z_k = \Gamma \circ D(z_{k+1}) \lneqq z_{k+1} \quad \text{for all } k \geq 0. \tag{13}$$

One can easily check that this sequence is unbounded in every component (as in the proof of Theorem 13).

The rest of the construction is similar to the construction of σ_s, but it is technical to make σ strictly increasing. This part is omitted for reasons of space, but can be found in Dashkovskiy et al. (2006a). □

Remark 15. The functions $\sigma \in \mathcal{K}_\infty^n$ that we constructed in Theorems 13 and 14 are possibly not smooth on a discrete set in $]0,\infty[$. Nevertheless, for each $i = 1,\dots,n$, the derivative σ_i' of σ_i is positive, except on this discrete set. This in particular implies $(\tilde{\sigma}_i^{-1})'(t) > 0$ for almost all $t > 0$ and $i = 1,\dots,n$.

Proof of Theorem 5. Just combine the statements of Proposition 7 and Theorem 14:

Recall that $D \circ \Gamma \not\geq \text{id}$ if and only if $\Gamma \circ D \not\geq \text{id}$. We may always decompose D into two diagonal operators D_1, D_2, see Dashkovskiy et al. (2006a), such that $D_1 \circ D_2 = D$, whereby D_1, D_2 are also of the form $\text{diag}_n(\text{id} + \alpha_i)$, $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2$.

So we have $D_1 \circ \Gamma \circ D_2 \not\geq \text{id}$, which we write $\tilde{\Gamma} \circ D_2 \not\geq \text{id}$. Now apply Theorem 14 to obtain a $\mathcal{K}_\infty^n$-function σ, satisfying

$$D_1 \circ \Gamma(\sigma(t)) = \tilde{\Gamma}(\sigma(t)) < \sigma(t), \text{for all } t > 0.$$

It remains to apply Proposition 7. □

5. CONCLUSIONS

We have constructed a nonsmooth ISS Lyapunov function for strongly connected networks of ISS systems. The resulting ISS Lyapunov function is a combination of the ISS Lyapunov functions of the subsystems. The path in $\mathbb{R}_+^n$ required for this combination was explicitly constructed for strongly connected networks.

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