Application of small gain type theorems in logistics of autonomous processes

S. Dashkovskiy B. Rüffer F. Wirth

Zentrum für Technomathematik, Universität Bremen

Abstract

In this paper we consider stability of logistic networks. We give a stability criterion for a general situation and show how it can be applied in special cases. For this purpose two examples are considered.

1 Introduction

The control of one processing machine or a small plant of several machines can be performed by one (central) control unite. A control can be designed in such a way that a production process is stable in the sense that the number of parts waiting in the buffer of the machine to be processed remains bounded. In case of a plant we can speak of the number of orders to be complete. Such number describes the state of a system. In case of large systems a centralized control becomes unflexible and may be even impossible. One of the possible solutions is the introduction of autonomous control [10], i.e., to allow single parts or machines to make decisions. Modelling of some simple scenarios with autonomous control by means of differential equations have been given in [2], [11]. Due to globalization many enterprizes begin to influence each other, to cooperate or merge together. Large logistic networks appear in this way. The question arises, under which conditions such networks are stable. Since the behavior of logistic processes is often nonlinear a suitable stability notion is the input-to-state stability (ISS) defined below. In this paper we will give a stability criterion for such logistic processes modelled by differential equations. Complex logistic processes are often described and modelled with help of graphs. A nodes of a graph may represent a processing machine or a plant in case of production logistics as well as a traffic junctions or a warehouse in case of transport logistics. The edges of the graph describe the connections or relations between the nodes. The stability property is very important for the design of logistic networks. This property ensures that the state of the whole system remains bounded for bounded external inputs. There are several stability criteria of a small gain type in the literature in the last decade [9], [8], [4]. Here we going to show how such theorems can be applied for the investigation of stability properties of logistic

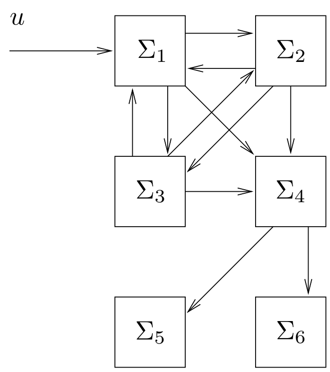

Figure 1: Graph structure of the logistic network

networks with autonomous control. For this purpose we consider two examples to show the essence and the method of usage of the small gain criteria. Some simulation results will be also presented.

2 Motivating example

Let us consider a network of car production. The first node Σ_1 receives the orders of customers. Depending on the customer wishes the order is then forwarded to a plant Σ_4 where the car will be completed or the preparatory steps in Σ_2 and/or Σ_3 are needed. The car is then supplied to one of the distribution centers Σ_5 or Σ_6. The corresponding graph of the network is given in Figure 1. The state of a plant is often model by differential equations [1], [7]. Let the following system on describe the evolution of states of the nodes.

$$\dot{x}_1 = u - \frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} \tag{1}$$

$$\dot{x}_2 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_3, c_3x_3\} - \min\{b_2, c_2x_2\} \tag{2}$$

$$\dot{x}_3 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_2, c_2x_2\} - \min\{b_3, c_3x_3\} \tag{3}$$

$$\dot{x}_4 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_2, c_2x_2\} + \min\{b_3, c_3x_3\} - \min\{b_4, c_4x_4\} \tag{4}$$

$$\dot{x}_5 = \frac{1}{2}\min\{b_4, c_4x_4\} - c_5x_5 \tag{5}$$

$$\dot{x}_6 = \frac{1}{2}\min\{b_4, c_4x_4\} - c_6x_6 \tag{6}$$

One can check that each subsystem is ISS using ISS Lyapunov functions $V_i(x_i) = x_i$, see [6] and [5], where the gains are also found. However the interconnection of these systems is not always stable. The small gain condition can be applied for these systems. It imposes certain restrictions on the constants a, b_i, c_i in the equations. See [5] for details. A numerical procedure was developed there to check the small gain condition locally. The procedure was applied for the stability investigation of the system (1-6). The motivation for influx and outflux terms is similar as in the example with two nodes, which we consider in detail in the next section.

3 Feedback loop as a two nodes network

We consider two stations processing parts coming from outside to the first station and forwarded to the second one after being processed there. After the processing on the second station the parts leave the network. The arrival process of parts is described in terms of the time varying arrival rate function $u = u(t)$ which can be considered as a result of another autonomous process. There are queues of length $x_1(t)$ and $x_2(t)$ in front of each station. The stations are considered to be self-controlled, i.e., able to increase or decrease their processing rates depending on certain circumstances, as for example current arrival rate, own queue length or the queue length of the neighbor. Such a system can be modelled with help of differential equations. Let be $b_1 = b_1(t, u, x_1, x_2)$ and $b_2 = b_2(t, u, x_1, x_2)$ the processing rate of the corresponding station. Then the current state of the system, i.e., the queue lengths, is then given as the solution of the following differential equations

$$\dot{x}_1 = u - b_1, \tag{7}$$

$$\dot{x}_2 = b_1 - b_2. \tag{8}$$

For our limited purpose it is enough to consider such a feedback loop. However it can be considered as a part of a more complex network.

3.1 Interpretations

Here we give some possible interpretations of the scenario. Consider a container ship being offloaded at a container terminal. The autonomous containers on the ship arrive to the offloading facility (the first station) in a certain rate u determined by their internal rules. The offloading facility consists of several cranes, each of them has several operating regimes. For example if the queue is short not all of them need to be active. If the queue becomes longer they are able to switch themselves to a faster regime. A desired processing rate can be achieved in this way. After offloading a container arrives to the customs clearance (the second station) and waits there to be processed. Since the space for this second queue is limited the offloading facility takes into account the length of the second queue and its processing rate can be taken in the form

$$b_1 = \frac{ax_1 + b\sqrt{x_1}}{1 + x_2}, \tag{9}$$

where a and b are positive constants. For vanishing x_2, i.e., for empty queue at customs, the processing rate is essentially proportional to x_1 if $x_1 >> 1$ and to $\sqrt{x_1}$ if $x_1 << 1$. With growth of the second queue x_2 this processing rate is reduced by the factor $(1 + x_2)$. The customs has several gates for container clearance with varying number of service personal. It uses only one gate if its queue is short and opens further gates when the queue grows. Let the processing rate of customs be proportional to the length of its own queue, i.e.,

$$b_2 = cx_2, \tag{10}$$

with a positive constant c. In this case the system (7-8) becomes

$$\dot{x}_1 = u - \frac{ax_1 + b\sqrt{x_1}}{1 + x_2} \tag{11}$$

$$\dot{x}_2 = \frac{ax_1 + b\sqrt{x_1}}{1 + x_2} - cx_2 \tag{12}$$

3.2 State equation and stability of the queues

The state equation (11-12) describes the evolution of the queues in dependence of the inflow u and the initial conditions. We assume that each subsystem is ISS. This means that there exists an ISS-Lyapunov function V_1 for each of them, i.e., for some $\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22}, \gamma$ of class $\mathcal{K}_\infty$, χ_1, χ_2 of class $\mathcal{K}$ and α_1, α_2 positive-definite functions it holds

$$\psi_{1i}(|x_1|) \leq V_i(x_1) \leq \psi_{2i}(|x_1|), \quad \forall x_1 \in \mathbb{R}^n, \ i = 1, 2, \tag{13}$$

$$|x_1| \geq \max\{\chi_1(|x_2|), \gamma(u)\} \ \Rightarrow \ \frac{dV_1}{dx_1}(x_1)\Big(u - \frac{ax_1 + \sqrt{x_1}}{1 + x_2}\Big) \leq -\alpha_1(|x_1|) \tag{14}$$

$$|x_2| \geq \chi_2(|x_1|) \ \Rightarrow \ \frac{dV_2}{dx_2}(x_2)\Big(\frac{ax_1 + \sqrt{x_1}}{1 + x_2} - cx_2\Big) \leq -\alpha_2(|x_2|) \tag{15}$$

Remark 1. *Note than the solution of (11) is nonnegative for any initial condition $x_1(0) = x_1^0 \geq 0$. This follows from $u \geq 0$ and any time when x_1 reaches zero $\dot{x}_1 = u \geq 0$ holds true. In the same way the solution of (12) is also nonnegative for any initial condition $x_2(0) = x_2^0 \geq 0$. This is in agrement with the notion of queue length, which cannot be negative. Hence in the following we will write x_1 and x_2 instead of $|x_1|$ and $|x_2|$ respectively.*

The small gain theorem in this case reads as follows

Theorem 2. *Let V_1 and V_2 satisfy (13-15) and assume that*

$$\chi_1 \circ \chi_2(s) < s, \tag{16}$$

then the interconnection (11-12) is ISS with an ISS-Lyapunov function given by

$$V(x_1, x_2) = \max\{\sigma(V_1(x_1)), V_2(x_2)\}, \tag{17}$$

where σ is a smooth class $\mathcal{K}_\infty$ function with

$$\chi_2 < \sigma < \chi_1^{-1}.$$

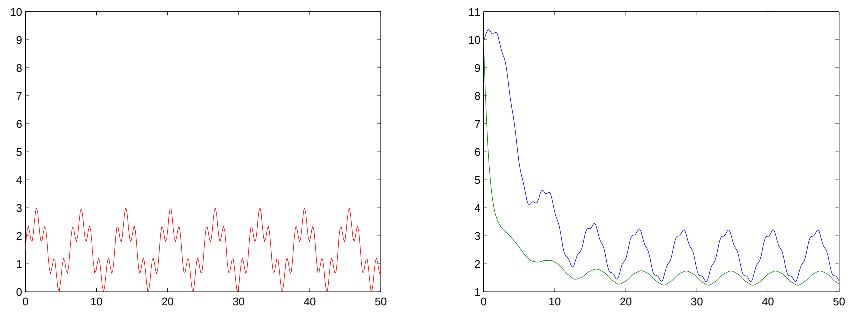

Figure 2: Input u and the queues x_1, x_2

We consider $V_1(x_1) = |x_1|$ and $V_2(x_2) = |x_2|$. These functions are not smooth in zero, however one can change both in an arbitrary small neighborhood of zero to be smooth. We will not go onto this technical details and will work with them as with smooth functions. For nonnegative arguments we have then

$$\frac{dV_1(x_1)}{dx_1} = \frac{dV_2(x_2)}{dx_2} = 1.$$

Let $\gamma(u) := u^2/c_u^2$ and $\chi_1(x_2) := x_2^2/c_1^2$, then from $x_1 > \chi_1(x_2)$ and $x_1 > \gamma(u)$ follows $u < \gamma^{-1}(x_1) = c_u\sqrt{x_1}$ and $x_2 < \chi_1^{-1}(x_1) = c_1\sqrt{x_1}$, i.e.,

$$u - \frac{ax_1 + b\sqrt{x_1}}{1 + x_2} < \gamma^{-1}(x_1) - \frac{ax_1 + b\sqrt{x_1}}{1 + \chi_1^{-1}(x_1)}$$

$$= \frac{c_u\sqrt{x_1} + c_u\sqrt{x_1}c_1\sqrt{x_1} - ax_1 - b\sqrt{x_1}}{1 + c_1\sqrt{x_1}} = -\frac{(a - c_uc_1)x_1 + (b - c_u)\sqrt{x_1}}{1 + c_1\sqrt{x_1}}$$

and $\frac{(a-c_uc_1)x_1+(b-c_u)\sqrt{x_1}}{1+c_1\sqrt{x_1}} =: \alpha_1(x_1)$ is a positive-definite function if $a > c_uc_1$ and $b > c_u$. This shows that (13) and (14) holds true, i.e., V_1 is an ISS-Lyapunov function and the equation (11) is ISS. Now consider equation (12) and let $\chi_2(x_2) := \sqrt{x_2}/\sqrt{c_2}$, then from $x_2 > \chi_2(x_1)$ follows $x_1 < \chi^{-1}(x_2) = c_2x_2^2$ and

$$\frac{ax_1 + b\sqrt{x_1}}{1 + x_2} - cx_2 < \frac{ac_2x_2^2 + b\sqrt{c_2}x_2 - cx_2 - cx_2^2}{1 + x_2} = -\frac{(c - ac_2)x_2^2 + (c - b\sqrt{c_2})x_2}{1 + x_2}$$

where $\frac{(c-ac_2)x_2^2+(c-b\sqrt{c_2})x_2}{1+x_2} =: \alpha_2(x_2)$ is a positive-definite function if $c > ac_2$ and $c > b\sqrt{c_2}$. Now we see that the small gain condition reads as

$$\chi_1 \circ \chi_2(s) < s \quad \Leftrightarrow \quad c_2c_1^2 > 1$$

and the network is stable if for the given parameters a, b, c of the system there exist positive c_u, c_1, c_2 such that $c_u < b$, $c_1 < a/c_u$, $c_2 < c/a$, $\sqrt{c_2} < c/b$ and $c_2c_1^2 > 1$. We can always take $c_2 < \min\{c/a, c^2/a^2\}$, $c_1 > 1/\sqrt{c_2}$ and $c_u < \min\{b, a/c_1\}$. Hence this network is input to state stable. The simulated solution of (11-12) with the input $u(t) = 1.5 + \sin(t) + \sin(5t)/2$ and the initial conditions $x_1(0) = 10, \quad x_2(0) = 10$ is presented on the Figure 2 We see that the solution remains bounded in this case.

4 Conclusions

The most important for the logistics is the knowledge of the long term behavior of the state of a system. Hence stability of a logistic networks is an important property which guarantees that the state remains bounded. A general criterion for the input-to-state stability was proposed. We have discussed how it can be applied in some special cases. In general it yields some restriction on the parameters of a system and can be checked numerically.

5 Acknowledgments

This research was supported by the German Research Foundation (DFG) as part of the Collaborative Research Center 637 "Autonomous Cooperating Logistic Processes".

References

[1] D. Armbruster, D. E. Marthaler, C. Ringhofer, K. Kempf, and T. Jo. A continuum model for a re-entrant factory. *Operations Research*, 54(5):933–950, 2006.

[2] S. Dachkovski, F. Wirth, and T. Jagalski. Autonomous control in shop floor logistics: Analytic models. In G. Chryssolouris and D. Mourtzis, editors, *Manufacturing, Modelling, Management and Control 2004*, Amsterdam, NL, 2005. Elsevier Science Ltd.

[3] S. Dashkovskiy, B. Rüffer, and F. Wirth. A small-gain type stability criterion for large scale networks of ISS systems. *Proc. of 44th IEEE Conference on Decision and Control and European Control Conference ECC 2005*, pages 5633–5638, 2005.

[4] S. Dashkovskiy, B. Rüffer, and F. Wirth. An ISS small-gain theorem for general networks. *Mathematics of Control, Signals, and Systems*, 19:93–122, 2007.

[5] S. Dashkovskiy, B. Rüffer, and F. Wirth. Numerical verification of local input-to-state stability for large networks. In *Proc. 46th IEEE Conf. Decision and Control, CDC 2007*, New Orleans, LA, December 2007. IEEE. submitted.

[6] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS Lyapunov function for networks of ISS systems. In *Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems (MTNS), Kyoto, Japan*, pages 77–82, July 24-28 2006.

[7] D. Helbing, L. S., W. U., and B. T. Network-induced oscillatory behavior in material flow networks and irregular business cycles. *Physical Review E*, 70(056118):1–6, 2004.

[8] Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica J. IFAC*, 32(8):1211–1215, 1996.

[9] Z.-P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Math. Control Signals Systems*, 7(2):95–120, 1994.

[10] B. Scholz-Reiter, T. Jagalski, and C. de Beer. Selbststeuerung logistischer Prozesse in Produktionsnetzen. *Industrie Management*, 23(1):19–22, 2007.

[11] B. Scholz-Reiter, F. Wirth, M. Freitag, S. Dashkovskiy, T. Jagalski, C. de Beer, and B. Rüffer. Mathematical models of autonomous logistic processes. In M. Hülsmann and K. Windt, editors, *Understanding Autonomous Cooperation & Control in Logistics : The Impact on Management, Information and Communication and Material Flow*, Berlin, 2007. Springer.

[12] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34(4):435–443, 1989.

6 Appendix: Definitions and known results

Let a logistic network be given. Consider equation describing dynamics of the state $x_i \in \mathbb{R}^{N_i}$ of the i-th node depending on the input $u_i \in \mathbb{R}^{M_i}$ and states of other nodes $x_j,\ j \neq i$

$$\dot{x}_i = f_i(x_1, \ldots, x_n, u_i), \quad i = 1, \ldots, n, \tag{18}$$

$f_i : \mathbb{R}^{\sum_j N_j + M_i} \to \mathbb{R}^{N_i}$, $i = 1, \ldots, n$. The system of these equations can be written in the form

$$\dot{x} = f(x, u) \tag{19}$$

with $x^T = (x_1^T, \ldots, x_n^T) \in \mathbb{R}^N$, $N = \sum_{i=1}^n N_i$, $u^T = (u_1^T, \ldots, u_n^T)$, $f(x,u)^T = (f_1(x,u_1)^T, \ldots, f_n(x,u_n)^T)$. Let $\mathbb{R}_+$ denote the interval $[0, \infty)$ and $\mathbb{R}_+^n$ be the positive orthant in $\mathbb{R}^n$. For any $a, b \in \mathbb{R}_+^n$ let $a < b$ and $a \leq b$ mean $a_i < b_i$ and $a_i \leq b_i$ for all $i = 1, \ldots, n$ respectively. Recall the definition of comparison functions.

Definition 3. *(i) A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. (ii) A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is non-increasing and tends to zero at infinity.*

The concept of input-to-state stability (ISS) has been first introduced in [12].

Definition 4. *System (19) is* input-to-state stable*, if there exists a $\gamma \in \mathcal{K}_\infty$, and a $\beta \in \mathcal{KL}$, such that for all $\xi \in \mathbb{R}_+^n$, $u \in L_\infty$*

$$\|x(t, \xi, u)\| \leq \beta(\|\xi\|, t) + \gamma(\|u\|_\infty) \quad \forall\, t \leq 0, \tag{20}$$

in this case γ is called gain.

It is known that ISS defined in this way is equivalent to the existence of an ISS-Lyapunov function.

Definition 5. *A smooth function $V : \mathbb{R}^N \to \mathbb{R}_+$ is an* ISS-Lyapunov function *of* (19) *if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}_\infty$, and a positive definite function α such that*

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^N, \tag{21}$$

$$V(x) \geq \chi(|u|) \Longrightarrow \nabla V(x) \cdot f(x, u) \leq -\alpha(V(x)), \tag{22}$$

for all $\xi \in \mathbb{R}^n_+$, $u \in L_\infty$. Function χ in then called Lyapunov-gain.

Subsystem (18) is ISS, provided there exist $\gamma_{ij}, \gamma_i \in \mathcal{K}_\infty$, and a $\beta_i \in \mathcal{KL}$, such that for all $\xi_i \in \mathbb{R}^n_+$, $u_i \in L_\infty$

$$\|x_i(t, \xi_i, x_j : j \neq i, u_i)\| \leq \max\{\beta_i(\|\xi_i\|, t), \max_{j \neq i} \gamma_{ij}(\|x_j\|_\infty), \gamma_i(\|u_i\|_\infty)\} \quad \forall t \geq 0, \tag{23}$$

If all n subsystems are ISS then these estimates give rise to a *gain matrix*

$$\Gamma = (\gamma_{ij})_{i,j=1}^n, \text{ with } \gamma_{ij} \in \mathcal{K}_\infty \text{ or } \gamma_{ij} \equiv 0, \tag{24}$$

where we use the convention $\gamma_{ii} \equiv 0$ for $i = 1, \dots, n$. The gain matrix Γ defines a monotone operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by $\Gamma(s)_i := \max_j \gamma_{ij}(s_j)$ for $s \in \mathbb{R}^n_+$. The global small gain condition assuring the ISS property for an interconnection of ISS subsystems was derived in [3]. An alternative proof has been given in [4]. We quote the following result from these papers.

Theorem 6 (global small-gain theorem for networks)**.** *Consider system (19) and suppose that each subsystem (18) is ISS, i.e., condition (23) holds for all $\xi_i \in \mathbb{R}^n_+$, $u_i \in L_\infty$, $i = 1, \dots, n$. Let Γ be given by (24). If there exists an $\alpha \in \mathcal{K}_\infty$, such that*

$$(\Gamma \circ D)(s) \not\geq s, \qquad \forall s \in \mathbb{R}^n_+ \setminus \{0\}, \tag{25}$$

with $D = \operatorname{diag}_n(\mathrm{id} + \alpha)$ then the system (19) is ISS from u to x.