Stability of autonomous vehicle formations using an ISS small-gain theorem for networks

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We consider a formation of vehicles moving on the two dimensional plane. The movement of each vehicle is described by a system of ordinary differential equations with inputs. The formation is maintained using autonomous controls that are designed to maintain fixed relative distances and orientations between vehicles. Moreover this formation should track a given trajectory on the plane. The vehicles can measure the relative distances and angles to their neighbors. These values are the inputs from one system to another. With the help of a general ISS small-gain theorem for networks we will show that the dynamics of such a formation is stable for the given controls. The notion of local input-to- state stability (local ISS) will be used for this purpose.

1 Input-to-state stability

In [7, 8, 6] the concept of input-to-state stability (ISS) [5] has been proposed under the name *Leader-to-formation stability* (LFS) to assess stability properties of formations of vehicles moving on the plane. Building on results in [6] we give a more general result on formation stability, which allows in contrast additional *looking back* feedback loops. More precisely, in our context it becomes possible to have cycles in the interconnection graph of the input-to-state stable error dynamics.

Consider the following set of n interconnected systems

$$\Sigma_i : \dot{x}_i = f_i(x_1, \dots, x_n, u), \qquad x_j \in \mathbb{R}^{N_j}, u \in \mathbb{R}^M, \qquad i = 1, \dots, n. \tag{1}$$

System Σ_i is ISS if there exists a loc. Lipschitz function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}$, a positive definite function α_i, functions $\psi_{i1}, \psi_{2i} \in \mathcal{K}_\infty$ and functions χ_{ij} which are either of class $\mathcal{K}_\infty$ or zero, such that

$$\psi_{i1}(\|x_i\|) \le V_i(x_i) \le \psi_{i2}(\|x_i\|),$$

and, for (Lebesgue) almost all x_i,

$$V_i(x_i) \ge \sum_{j \ne i} \chi_{ij}(V_j(x_j)) + \chi_i(\|u\|) \implies \nabla V_i(x_i) \cdot f_i(x, u) \le -\alpha_i(V_i(x_i)). \tag{2}$$

The system is called locally ISS if there exists a constant $C_i > 0$ such that (2) holds for all $\|x_i\|, \|x_j\|, \|u\| < C_i$.

From [1] we have a general small-gain theorem for networks of ISS systems. A corresponding local Lyapunov version is (cf. [2, 3, 4])

Theorem 1.1 *Assume a network of interconnected systems (1) be given and is (locally) ISS, i.e., satisfies (2). If there exist functions $\sigma_i \in \mathcal{K}_\infty, \phi \in \mathcal{K}_\infty$ (and some $R > 0$), so that each subsystem Σ_i, $i = 1, \dots, n$, satisfies*

$$\sum_{j \ne i} \chi_{ij}(\sigma_j(r)) + \chi_i(\phi(r)) < \sigma_i(r) \tag{*}$$

for all $r > 0$ (for all $0 < r < R$), then $V(x) = \max_i \sigma_i^{-1}(V_i(x_i))$ is a nonsmooth (local) ISS Lyapunov function for the system $\dot{x} = f(x, u)$.

Condition (*) is effectively the generalized small gain condition. Key to this result is of course the existence of such functions σ_i, which can be guaranteed if a small gain condition is satisfied [2, 3].

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2 Vehicle formation control and ISS

We consider a group of n vehicles. The kinematic model of each vehicle is given by $\dot{x}_i = v_i \cos\theta_i$, $\dot{y}_i = v_i \sin\theta_i$, and $\dot{\theta}_i = \omega_i$, where (x_i, y_i) is the absolute position and θ_i the orientation. The translational and rotational velocities are v_i and ω_i, respectively. The relationship between leader i and follower j is expressed in terms of the separation distance l_{ij} and relative bearing ψ_{ij}. We consider the deviation of these variables from specification parameters l_{ij}^d, ψ_{ij}^d, i.e., our states are $\tilde{l}_{ij} = l_{ij}^d - l_{ij}$, and $\tilde{\psi}_{ij} = \psi_{ij}^d - \psi_{ij}$, and we denote $\tilde{z}_{ij} = (\tilde{l}_{ij}, \tilde{\psi}_{ij})^\top$. Let $\phi_{ij} = \theta_i - \theta_j$ and denote by d the distance from the wheel axis to a reference point. Then the leader-follower dynamics takes the form

$$\begin{pmatrix} \dot{\tilde{l}}_{ij} \\ \dot{\tilde{\psi}}_{ij} \\ \dot{\phi}_{ij} \end{pmatrix} = \begin{pmatrix} \cos\psi_{ij} & 0 \\ \frac{-\sin\psi_{ij}}{l_{ij}} & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_i \\ \omega_i \end{pmatrix} - \begin{pmatrix} \cos(\phi_{ij}+\psi_{ij}) & d\sin(\phi_{ij}+\psi_{ij}) \\ \frac{-\sin(\phi_{ij}+\psi_{ij})}{l_{ij}} & \frac{d\cos(\phi_{ij}+\psi_{ij})}{l_{ij}} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_j \\ \omega_j \end{pmatrix}.$$

Denote the upper 2×2 block of the left matrix by A_{ij} the upper 2×2 block of the right matrix by B_{ij}. Then, using input-output feedback linearization, it has been shown [6] that the control input $\begin{pmatrix} v_j \\ \omega_j \end{pmatrix} = B_{ij}^{-1} K^j \begin{pmatrix} \tilde{l}_{ij} \\ \tilde{\psi}_{ij} \end{pmatrix}$ renders the error dynamics $\dot{\tilde{z}}_{ij} = A_{ij} \begin{pmatrix} v_i \\ \omega_i \end{pmatrix} - B_{ij} \begin{pmatrix} v_j \\ \omega_j \end{pmatrix}$ input to state stable from $\tilde{z}_{ki}$ to $\tilde{z}_{ij}$, where k is the number of the vehicle in front of vehicle i.

We extend the velocity control for the first vehicle in our group in Figure 1 to obtain

$$\begin{aligned} \begin{pmatrix} v_1 \\ \omega_1 \end{pmatrix} &= B_{P1}^{-1} K^{1f} \tilde{z}_{P1} + B_{12}^{-1} K^{1bl} \tilde{z}_{12} + B_{13}^{-1} K^{1br} \tilde{z}_{13}, \\ \begin{pmatrix} v_2 \\ \omega_2 \end{pmatrix} &= B_{12}^{-1} K^{2f} \tilde{z}_{12}, \\ \begin{pmatrix} v_3 \\ \omega_3 \end{pmatrix} &= B_{13}^{-1} K^{2f} \tilde{z}_{13}. \end{aligned}$$

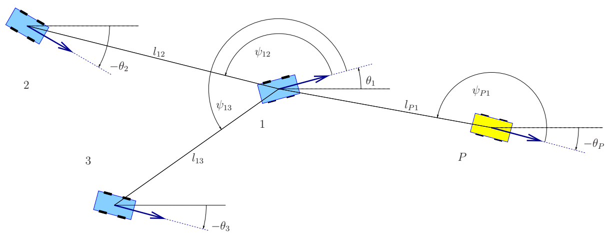

Fig.1 A formation of three vehicles (1–3) following a phantom leader (P).

With the above feedback control for the formation given in Fig. 1 and the Lyapunov function candidates $V_j(\tilde{z}_{ij}) = \frac{1}{2}\tilde{z}_{ij}^\top (K^{jf})^{-1} \tilde{z}_{ij}$ we obtain a set of nonlinear gains $\gamma_{P1,12}, \gamma_{P1,13}, \gamma_{12,P1}, \gamma_{12,13}, \gamma_{13,P1}, \gamma_{13,12}$. For suitable choices of parameters these gains indeed satisfy condition (*), so that the formation error is locally input-to-state stable with respect to external inputs.

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References

[1] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS small gain theorem for general networks. *Mathematics of Control, Signals, and Systems (MCSS)*, 19(2):93–122, 2007.
[2] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. Construction of ISS Lyapunov functions for networks. Technical Report 06-06, Zentrum für Technomathematik, University of Bremen, Juli 2006.
[3] S. Dashkovskiy, B S. Rüffer, and F. R. Wirth. An ISS Lyapunov function for networks of ISS systems. In *Proc. of the 17th International Symposium on Mathematical Theory of Networks and Systems (MTNS), Kyoto, Japan*, 77–82, July 24-28 2006.
[4] B. S. Rüffer. *Monotone dynamical systems, graphs, and stability of large-scale interconnected systems*. PhD thesis, Fachbereich 3, Mathematik und Informatik, Universität Bremen, Germany, 2007. Available online at http://nbn-resolving.de/urn:nbn:de:gbv:46-diss000109058.
[5] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34(4):435–443, 1989.
[6] H. G. Tanner, G. J. Pappas, and V. Kumar. Leader-to-formation stability. *IEEE Transactions on Robotics and Automation*, 20(3):443–455, 2004.
[7] H.G. Tanner, V. Kumar, and G.J. Pappas. The effect of feedback and feedforward on formation ISS. In *Proceedings of the ICRA'02. IEEE International Conference on Robotics and Automation*, 4:3448–3453, 2002.
[8] H.G. Tanner, V. Kumar, and G.J. Pappas. Stability properties of interconnected vehicles. In *Proc. of the Fifteenth International Symposium on Mathematical Theory of Networks and Systems, University of Notre Dame, August 12-16*, pages (12), paper number 4615–2, 2002.