Small gain theorems for large scale systems and construction of ISS Lyapunov functions*

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Abstract— **We consider interconnections of n nonlinear subsystems in the input-to-state stability (ISS) framework. For each subsystem an ISS Lyapunov function is given that treats the other subsystems as independent inputs. A gain matrix is used to encode the mutual dependencies of the systems in the network. Under a small gain assumption on the monotone operator induced by the gain matrix, a locally Lipschitz continuous ISS Lyapunov function is obtained constructively for the entire network by appropriately scaling the individual Lyapunov functions for the subsystems.**

Index Terms— **Nonlinear systems, input-to-state stability, interconnected systems, large scale systems, Lipschitz ISS Lyapunov function, small gain condition**

I. Introduction

In many applications large scale systems are obtained through the interconnection of a number of smaller components. The stability analysis of such interconnected systems can be a difficult task especially in the case of a large number of subsystems, arbitrary interconnection topologies, and nonlinear subsystems.

One of the earliest tools in the stability analysis of feedback interconnections of nonlinear systems are small gain theorems. Such results have been obtained by many authors starting with [39], but see also [2] for a recent account of the developments in this area.

Small gain theorems for large scale systems have been developed, e.g., in [30], [38], [24], [14], with the common restriction that gains describing the interconnection are essentially linear.

With the introduction of the concept of input-to-state stability (ISS) in [31], it has become a common approach to consider gains as nonlinear functions of the norm of the input. In this nonlinear case small gain results have been derived first for the interconnection of two systems in [17], [35], [18] followed by [23], [36], [4], [5] for many systems.

Other general ISS small-gain theorems are given in [7], [6], [5], [20], [21] [11], [16] [22], [15], [37], [19]. In some of these references ISS of subsystems is studied in the maximization framework leading to a small gain condition in the 'cycle formulation'. It has been noted already in [7] that in the maximum case the cycle condition becomes a special case of the operator condition examined here. All these conditions can be interpreted as an ISS requirement for an associated comparison system [28], [27] and there exist numerical methods to verify them [29], [13].

We present a unified framework for different formulations of input-to-state stability with respect to several inputs, which we call *monotone aggregation functions*. In particular, this leads to a unified perspective on the sufficiency criteria in different formulations.

As our main contribution in this paper we present sufficient conditions for the existence of an ISS Lyapunov function for a system obtained as the interconnection of many subsystems. The results are of interest in two ways. First, it is shown that a small gain condition is sufficient for input-to-state stability of the large scale system in the Lyapunov formulation. Secondly, an explicit formula for an overall Lyapunov function is given. As the dimensions of the subsystems are essentially lower than the dimension of their interconnection, finding Lyapunov functions for them may be an easier task than for the whole system.

The paper is organized as follows. The next section introduces the necessary notation and basic definitions, in particular the notion of monotone aggregation functions (MAFs) and different formulations of ISS. In Section III we introduce small gain conditions given in terms of monotone operators that naturally appear in the definition of ISS. Section IV contains the main results, namely the existence of a vector scaling function σ and the construction of an ISS Lyapunov function based on this function σ. In this section we concentrate on strongly connected networks which are easier to deal with from a technical point of view. The existence of σ is given in Section V to postpone the topological considerations until after applications to interconnected ISS systems have been considered. Here we also show to address simply connected networks by resorting to a result from [26]. Section VI concludes the paper.

This work does not contain examples and applications, but the interested reader can find these in, e.g., [8], [9]. All proofs have been omitted but can be found in [10].

II. Preliminaries

A. Notation and conventions

Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}^n$ the vector space of real column vectors of length n. We denote the set of nonnegative real numbers by $\mathbb{R}_+$ and $\mathbb{R}^n_+ := (\mathbb{R}_+)^n$ denotes the positive orthant in $\mathbb{R}^n$. On $\mathbb{R}^n_+$ the standard partial order is defined as follows. For vectors $v, w \in \mathbb{R}^n$ we denote

$$v \geq w :\iff v_i \geq w_i \text{ for } i = 1, \dots, n,$$
$$v > w :\iff v_i > w_i \text{ for } i = 1, \dots, n,$$
$$v \gneqq w :\iff v \geq w \text{ and } v \neq w.$$

The maximum of two vectors or matrices is to be understood component-wise. By $|\cdot|$ we denote the 1-norm on $\mathbb{R}^n$ and by S_r the induced sphere of radius r in $\mathbb{R}^n$ intersected with $\mathbb{R}^n_+$, which is an $(n-1)$-simplex. On $\mathbb{R}^n_+$ we denote by $\pi_I : \mathbb{R}^n_+ \to \mathbb{R}^{\#I}_+$ the *projection* of the coordinates in $\mathbb{R}^n_+$ corresponding to the indices in $I \subset \{1, \dots, n\}$ onto $\mathbb{R}^{\#I}$.

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The standard scalar product in $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$. By $U_\varepsilon(x)$ we denote the open ball of radius ε around x with respect to the Euclidean norm $\|\cdot\|$. The induced operator norm, i.e. the spectral norm, of matrices is also denoted by $\|\cdot\|$.

The space of measurable and essentially bounded functions is denoted by L^∞ with norm $\|\cdot\|_\infty$. To state the stability definitions that we are interested in three sets of comparison functions are used: $\mathcal{K} = \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+, \gamma$ is continuous, strictly increasing, and $\gamma(0) = 0\}$ and $\mathcal{K}_\infty = \{\gamma \in \mathcal{K} : \gamma$ is unbounded$\}$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$, if it is of class $\mathcal{K}$ in the first argument and strictly decreasing to zero in the second argument. We will call a function $V : \mathbb{R}^N \to \mathbb{R}_+$ *proper and positive definite* if there are $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that

$$\psi_1(\|x\|) \le V(x) \le \psi_2(\|x\|), \quad \forall x \in \mathbb{R}^N .$$

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is called *positive definite* if it is continuous and satisfies $\alpha(r) = 0$ if and only if $r = 0$.

B. Problem Statement

We consider a finite set of interconnected systems with state $x = \left(x_1^T, \ldots, x_n^T\right)^T$, where $x_i \in \mathbb{R}^{N_i}$, $i = 1, \ldots, n$ and $N := \sum N_i$. For $i = 1, \ldots, n$ the dynamics of the i-th subsystem is given by

$$\begin{aligned} \Sigma_i :\ & \dot{x}_i = f_i(x_1, \ldots, x_n, u), \\ & x \in \mathbb{R}^N,\ u \in \mathbb{R}^M,\ f_i : \mathbb{R}^{N+M} \to \mathbb{R}^{N_i} . \end{aligned} \qquad (1)$$

For each i we assume unique existence of solutions and forward completeness of Σ_i in the following sense. If we interpret the variables x_j, $j \ne i$, and u as unrestricted inputs, then this system is assumed to have a unique solution defined on $[0, \infty)$ for any given initial condition $x_i(0) \in \mathbb{R}^{N_i}$ and any L^∞-inputs $x_j : [0, \infty) \to \mathbb{R}^{N_j}$, $j \ne i$, and $u : [0, \infty) \to \mathbb{R}^M$. This can be guaranteed for instance by suitable Lipschitz and growth conditions on the f_i. It will be no restriction to assume that all systems have the same (augmented) external input u.

We write the interconnection of subsystems (1) as

$$\Sigma :\ \dot{x} = f(x, u), \quad f : \mathbb{R}^{N+M} \to \mathbb{R}^N. \qquad (2)$$

Associated to such a network is a directed graph, with vertices representing the subsystems and where the directed edges (i, j) correspond to inputs going from system j to system i. We will call the network strongly connected if its interconnection graph has the same property.

For networks of the type that has been just described we wish to construct Lyapunov functions as they are introduced now.

C. Stability

An appropriate stability notion to study nonlinear systems with inputs is input-to-state stability, introduced in [31]. The standard definition is as follows.

A forward complete system $\dot{x} = f(x, u)$ with $x \in \mathbb{R}^N, u \in \mathbb{R}^M$ is called input-to-state stable if there are $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that for all initial conditions $x_0 \in \mathbb{R}^N$ and all $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^M)$ we have

$$\|x(t; x_0, u(\cdot))\| \le \beta(\|x_0\|, t) + \gamma(\|u\|_\infty) .$$

It is known to be an equivalent requirement to ask for the existence of an ISS Lyapunov function [33]. These functions can be chosen to be smooth. For our purposes, however, it will be more convenient to have a broader class of functions available for the construction of a Lyapunov function. Thus we will call a function a *Lyapunov function candidate*, if the following assumption is met.

Assumption 1 *The function $V : \mathbb{R}^N \to \mathbb{R}_+$ is continuous, proper and positive definite and locally Lipschitz continuous on $\mathbb{R}^N \setminus \{0\}$.*

Note that by Rademacher's Theorem (e.g., [12, Theorem 5.8.6, p.281]) locally Lipschitz continuous functions on $\mathbb{R}^N \setminus \{0\}$ are differentiable almost everywhere in $\mathbb{R}^N$.

Definition 2 *We will call a function satisfying Assumption 1 an ISS Lyapunov function for $\dot{x} = f(x, u)$ if there exist $\gamma \in \mathcal{K}$ and a positive definite function α such that in all points of differentiability of V we have*

$$V(x) \ge \gamma(\|u\|) \implies \nabla V(x) f(x, u) \le -\alpha(\|x\|) .$$

ISS and ISS Lyapunov functions are related in the expected manner [33], [10]:

Theorem 3 *A system is ISS if and only if it admits an ISS Lyapunov function in the sense of Definition 2.*

D. Monotone aggregation

In this paper we concentrate on the construction of ISS Lyapunov functions for the interconnected system Σ. For a single subsystem (1) we wish to quantify the combined effect of the inputs x_j, $j \ne i$, and u on the evolution of the state x_i. It depends on the system under consideration how this combined effect can be expressed, through the sum of individual effects, using the maximum of individual effects or by other means. In order to be able to give a general treatment of this we introduce the notion of *monotone aggregation functions* (MAFs).

Definition 4 *A continuous function $\mu : \mathbb{R}^n_+ \to \mathbb{R}_+$ is called a monotone aggregation function if the following three properties hold*

- (M1) *positive definiteness: $\mu(s) \ge 0$ for all $s \in \mathbb{R}^n_+$ and $\mu(s) = 0$ if and only if $s = 0$;*
- (M2) *strict increase*[1]*: if $x < y$, then $\mu(x) < \mu(y)$;*
- (M3) *unboundedness: if $\|x\| \to \infty$ then $\mu(x) \to \infty$.*

The space of monotone aggregation functions is denoted by MAF_n and $\mu \in \mathrm{MAF}_n^m$ denotes a vector MAF, i.e., $\mu_i \in \mathrm{MAF}_n$, for $i = 1, \ldots, m$.

A direct consequence of (M2) and continuity is the weaker monotonicity property

(M2') *monotonicity: $x \le y \implies \mu(x) \le \mu(y)$.*

In [25], [26] MAFs have additionally been required to satisfy another property,

(M4) *subadditivity: $\mu(x + y) \le \mu(x) + \mu(y)$.*

Standard examples of monotone aggregation functions satisfying (M1)—(M4) are

$$\mu(s) = \sum_{i=1}^n s_i^l, \text{ where } l \ge 1, \text{ or}$$

$$\mu(s) = \max_{i=1,\ldots,n} s_i \text{ or}$$

$$\mu(s_1, s_2, s_3, s_4) = \max\{s_1, s_2\} + \max\{s_3, s_4\} .$$

On the other hand, the following function is not a MAF, since (M1) and (M3) are not satisfied; $\nu(s) = \prod_{i=1}^n s_i$.

Using this definition we can define a notion of ISS Lyapunov function for systems with multiple inputs. In this case Σ_i in (1) will have several gains γ_{ij} corresponding to the inputs x_j. For notational simplicity, we will include the

[1] Cf. Assumption (6), where for the purposes of this paper (M2) is further restricted.

gain $\gamma_{ii} \equiv 0$ throughout this paper. The following definition requires only Lipschitz continuity of the Lyapunov function.

Definition 5 *Assume that for each subsystem Σ_i given by* (1) *there is a function $V_i : \mathbb{R}^{N_j} \to \mathbb{R}_+$ satisfying Assumption 1.*

For $i = 1, \ldots, n$ the function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ is called an ISS Lyapunov function for Σ_i, if there exist $\mu_i \in \mathrm{MAF}_{n+1}, \gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}, j \neq i, \gamma_{iu} \in \mathcal{K} \cup \{0\}$, and a positive definite function α_i such that at all points of differentiability of V_i

$$V_i(x_i) \geq \mu_i\left(\gamma_{i1}(V_1(x_1)), \ldots, \gamma_{in}(V_n(x_n)), \gamma_{iu}(\|u\|)\right) \implies \nabla V_i(x_i) f_i(x, u) \leq -\alpha_i(\|x_i\|)\,. \tag{3}$$

The functions γ_{ij} and γ_{iu} are called ISS Lyapunov gains.

Several examples of ISS Lyapunov functions are given in the next section.

Let us call x_j, $j \neq i$, the *internal inputs* to Σ_i and u the *external input*. Note that the role of functions γ_{ij} and γ_{iu} is essentially to indicate whether there is any influence of different inputs on the corresponding state. In case f_i does not depend on x_j there is no influence of x_j on the state of Σ_i. In this case we define $\gamma_{ij} \equiv 0$, in particular always $\gamma_{ii} \equiv 0$. This allows us to collect the internal gains into a matrix

$$\Gamma := (\gamma_{ij})_{i,j=1,\ldots,n}\,. \tag{4}$$

If we add the external gains as the last column into this matrix then we denote it by $\overline{\Gamma}$. The function μ_i describes how the internal and external gains interactively enter in a common influence on x_i. The above definition motivates the introduction of the nonlinear map $\overline{\Gamma}_\mu \colon \mathbb{R}_+^{n+1} \to \mathbb{R}_+^n$,

$$\begin{bmatrix} s_1 \\ \vdots \\ s_n \\ r \end{bmatrix} \mapsto \begin{bmatrix} \mu_1(\gamma_{11}(s_1), \ldots, \gamma_{1n}(s_n), \gamma_{1u}(r)) \\ \vdots \\ \mu_n(\gamma_{n1}(s_1), \ldots, \gamma_{nn}(s_n), \gamma_{nu}(r)) \end{bmatrix}. \tag{5}$$

Similarly we define $\Gamma_\mu(s) := \overline{\Gamma}_\mu(s, 0)$. The matrices Γ and $\overline{\Gamma}$ are from now on referred to as *gain matrices*, Γ_μ and $\overline{\Gamma}_\mu$ as *gain operators*.

Remark 6 (general assumption) *Given $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ and $\mu \in \mathrm{MAF}^n$, we will from now on assume that Γ and μ are* compatible *in the following sense: For each $i = 1, \ldots, n$, let I_i denote the set of indices corresponding to the nonzero entries in the ith row of Γ. Then it is understood that also the restriction of μ_i to the indices I_i satisfies (M2), i.e.,*

$$\mu_i(x|_{I_i}) < \mu_i(y|_{I_i}) \quad \text{if} \quad x|_{I_i} < y|_{I_i}\,. \tag{6}$$

In particular we assume that the function

$$s \mapsto \mu(s_1, \ldots, s_n, 0), \quad s \in \mathbb{R}_+^n,$$

for $\mu \in \mathrm{MAF}_{n+1}$ satisfies (M2). Note that (M1) and (M3) are automatically satisfied.

E. Special case: Maximization

The case when the aggregation is the maximum, i.e., $\mu = \max$, is indeed a special case, since not only the small gain condition can be formulated in simpler manner, but also the path construction can be achieved without the need of the diagonal operator D as before.

A *cycle* in a matrix Γ is finite sequence of nonzero entries of Γ of the form

$$(\gamma_{i_1,i_2}, \gamma_{i_2,i_3}, \ldots, \gamma_{i_K,i_1}).$$

A cycle is called *subordinated* if $i_1 > \max\{i_2, \ldots, i_K\}$, and it is called a *contraction*, if

$$\gamma_{i_1,i_2} \circ \gamma_{i_2,i_3} \circ \ldots \circ \gamma_{i_K,i_1} < \mathrm{id}.$$

It is an easy exercise to show that when all subordinated cycles are contractions then already all cycles are contractions.

III. Monotone Operators and Generalized Small Gain Conditions

In Section II-D we saw that in the ISS context the mutual influence between subsystems (1) and the influence from external inputs to the subsystems can be quantified by the gain matrices Γ and $\overline{\Gamma}$ and gain operators Γ_μ and $\overline{\Gamma}_\mu$. The interconnection structure of the subsystems naturally leads to a weighted, directed graph, where the weights are the nonlinear gain functions, and the vertices are the subsystems. There is an edge from the vertex i to the vertex j if and only if there is an influence of the state x_i on the state x_j, i.e., there is a nonzero gain γ_{ji}.

Connectedness properties of the interconnection graph together with mapping properties of the gain operators will yield a generalized small gain condition. In essence we need a nonlinear version of a Perron vector for the construction of a Lyapunov function for the interconnected system. This will be made rigorous in the sequel. But first we introduce some further notation.

The adjacency matrix $A_\Gamma = (a_{ij})$ of a matrix $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ is defined by $a_{ij} = 0$ if $\gamma_{ij} \equiv 0$ and $a_{ij} = 1$ otherwise. Then $A_\Gamma = (a_{ij})$ is also the adjacency matrix of the graph representing an interconnection.

We say that a matrix Γ is *primitive, irreducible* or *reducible* if and only if A_Γ is primitive, irreducible or reducible, respectively. Recall (and see [1] for more on this subject) that a nonnegative matrix A is

- *primitive* if there exists a $k \geq 1$ such that A^k is positive;
- *irreducible* if for every pair (i, j) there exists a $k \geq 1$ such that the (i, j)th entry of A^k is positive; obviously, primitivity implies irreducibility;
- *reducible* if it is not irreducible.

A network or a graph is strongly connected if and only if the associated adjacency matrix is irreducible, see also [1].

For $\mathcal{K}_\infty$ functions $\alpha_1, \ldots, \alpha_n$ we define a diagonal operator $D : \mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$D(s) := (s_1 + \alpha_1(s_1), \ldots, s_n + \alpha_n(s_n))^T, \quad s \in \mathbb{R}_+^n. \tag{7}$$

For an operator $T : \mathbb{R}_+^n \to \mathbb{R}_+^n$, the condition $T \not\geq \mathrm{id}$ means that for all $s \neq 0$, $T(s) \not\geq s$. In words, at least one component of $T(s)$ has to be strictly less than the corresponding component of s.

Definition 7 (Small gain conditions) *Let a gain matrix Γ and a monotone aggregation μ be given. The operator Γ_μ is said to satisfy the small gain condition* (SGC)*, if*

$$\Gamma_\mu \not\geq id, \tag{SGC}$$

Furthermore, Γ_μ satisfies the strong small gain condition (sSGC)*, if there exists a D as in* (7) *such that*

$$D \circ \Gamma_\mu \not\geq id\,. \tag{sSGC}$$

It is not difficult to see that (sSGC) can equivalently be stated as

$$\Gamma_\mu \circ D \not\geq \mathrm{id}. \tag{sSGC'}$$

Also for (sSGC) or (sSGC') to hold it is sufficient to assume that the function $\alpha_1, \ldots, \alpha_n$ are all identical. This can be seen by defining $\alpha(s) := \min_i \alpha_i(s)$. We abbreviate this in writing $D = \mathrm{diag}(\mathrm{id} + \alpha)$ for some $\alpha \in \mathcal{K}_\infty$.

For maps $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ we define the following sets:

$$\Omega(T) := \{s \in \mathbb{R}^n_+ : T(s) < s\} = \bigcap_{i=1}^n \Omega_i(T), \text{ where}$$
$$\Omega_i(T) := \{s \in \mathbb{R}^n_+ : T(s)_i < s_i\}\,.$$

If no confusion arises we will omit the reference to T. Topological properties of the introduced sets are related to the small gain conditions (SGC), cf. also [4], [5], [26]. They will be used in the next section for the construction of an ISS Lyapunov function for the interconnection.

IV. Lyapunov functions

In this section we present the two main results of the paper. The first is a topological result on the existence of a jointly unbounded path in the set Ω, provided that Γ_μ satisfies the small gain condition. This path will be crucial in the construction of a Lyapunov function, which is the second main result of this section.

Definition 8 *A continuous path $\sigma\colon \mathbb{R}_+ \to \mathbb{R}^n_+$ with $\sigma_i \in \mathcal{K}_\infty$ for all i will be called an* Ω-path with respect to Γ_μ *if*

(i) *for each i, the function σ_i^{-1} is locally Lipschitz continuous on $(0,\infty)$;*

(ii) *for every compact set $K \subset (0,\infty)$ there are constants $0 < c < C$ such that for all $i = 1, \ldots, n$ and all points of differentiability of σ_i^{-1} we have*

$$0 < c \leq (\sigma_i^{-1})'(r) \leq C\,, \quad \forall r \in K; \tag{8}$$

(iii) *$\sigma(r) \in \Omega(\Gamma_\mu)$ for all $r > 0$, i.e.*

$$\Gamma_\mu(\sigma(r)) < \sigma(r)\,, \quad \forall r > 0\,.$$

Now we can state the first of our two main results, which regards the existence of Ω-paths.

Theorem 9 *Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ be a gain matrix and $\mu \in \mathrm{MAF}^n_n$. Assume that one of the following assumptions is satisfied*

(i) *Γ_μ is linear and the spectral radius of Γ_μ is less than one;*

(ii) *Γ is irreducible and $\Gamma_\mu \not\geq id$;*

(iii) *$\mu = \max$ and $\Gamma_\mu \not\geq id$; or, equivalently, $\mu = \max$ and all subordinated cycles of Γ are contractions;*

(iv) *alternatively assume that Γ_μ is bounded, i.e., $\Gamma \in \big((\mathcal{K} \setminus \mathcal{K}_\infty) \cup \{0\}\big)^{n\times n}$, and satisfies $\Gamma_\mu \not\geq id$.*

Then there exists an Ω-path σ with respect to Γ_μ.

For the proof see [10].

In addition to the above result, the existence of Ω-paths can also be asserted for reducible Γ and Γ with mixed, bounded and unbounded, class $\mathcal{K}$ entries, see Theorem 18 and Proposition 19, respectively. Only existence of an Ω-paths is shown there. An analytic construction is known only for the case $\mu = \max$ see, [21]. For a general case numerical constructions were developed in [29], [13].

Theorem 10 *Consider the interconnected system Σ given by* (1), (2) *where each of the subsystems Σ_i has an ISS Lyapunov function V_i, the corresponding gain matrix is given by* (4)*, and $\mu = (\mu_1, \ldots, \mu_n)^T$ is given by* (3)*. Assume there are an Ω-path σ with respect to Γ_μ and a function $\varphi \in \mathcal{K}_\infty$ such that*

$$\overline{\Gamma}_\mu(\sigma(r), \varphi(r)) < \sigma(r)\,, \quad \forall\ r > 0 \tag{9}$$

is satisfied, then an ISS Lyapunov function for the overall system is given by

$$V(x) = \max_{i=1,\ldots,n} \sigma_i^{-1}(V_i(x_i))\,. \tag{10}$$

In particular, for all points of differentiability of V we have the implication

$$\begin{aligned} V(x) \geq \max\{\varphi^{-1}(\gamma_{iu}(\|u\|)) \mid i = 1, \ldots n\} \\ \implies \nabla V(x) f(x,u) \leq -\alpha(\|x\|)\,, \end{aligned} \tag{11}$$

where α is a suitable positive definite function.

Note that by construction the Lyapunov function V is not smooth, even if the functions V_i for the subsystems are. This is why it is appropriate in this framework to consider Lipschitz continuous Lyapunov functions, which are differentiable almost everywhere.

In the absence of external inputs, ISS is the same as 0-GAS (cf. [32], [33], [34]). We note the following consequence in the case that only global asymptotic stability is of interest.

Corollary 11 *In the setting of Theorem 10, assume that the external inputs satisfy $u \equiv 0$ and that the network of interconnected systems is strongly connected. If $\Gamma_\mu \not\geq id$ then the network is 0-GAS.*

Remark 12 *At first sight it might seem that the previous corollary is stronger than [17, Cor. 2.1], as no robustness term D is needed in the assumptions. However, the result here is formulated for Lyapunov functions whereas the result in [17] is based on the trajectory formulation of ISS in summation form. The proof in the trajectory version essentially requires bounds on $(id - \Gamma_\mu)^{-1}$, which relies heavily on D unless $\mu = \max$, [17], [5], [25]. In contrast, for* 0*-GAS the D is not needed in the Lyapunov setting, because for irreducible Γ it is possible to construct the path σ without D by Theorem 9 (ii). As well D in not needed to prove ISS in case $\mu = \max$, however it is needed in a general case, for example when $\mu = \sum$. The difference between the SGC and sSGC especially in cases $\mu = \max$ and $\mu = \sum$ was studied in detail in [3].*

We now specialize the Theorem 10 to particular cases of interest. Namely, when the gain with respect to the external input u enters the ISS condition (i) additively, (ii) via maximization and (iii) as a factor.

Corollary 13 (Additive gain of external input u) *Consider the interconnected system Σ given by* (1), (2) *where each of the subsystems Σ_i has an ISS Lyapunov function V_i and the corresponding gain matrix is given by* (5)*. Assume that the ISS condition is additive in the gain of u, that is,*

$$\overline{\Gamma}_\mu(V_1(x_1), \ldots, V_n(x_n), \|u\|) = \Gamma_\mu(V_1(x_1), \ldots, V_n(x_n)) + \gamma_u(\|u\|)\,,$$

where $\gamma_u(\|u\|) = (\gamma_{1u}(\|u\|), \ldots, \gamma_{nu}(\|u\|))^T$. If Γ_μ is irreducible and if there exists an $\alpha \in \mathcal{K}_\infty$ such that for $D = \mathrm{diag}(id + \alpha)$ the gain operator Γ_μ satisfies the strong small gain condition

$$D \circ \Gamma_\mu(s) \not\geq s$$

then the interconnected system is ISS and an ISS Lyapunov function is given by (10)*, where $\sigma \in \mathcal{K}_\infty^n$ is an arbitrary Ω-path with respect to $D \circ \Gamma_\mu$.*

Corollary 14 (Maximization w.r.t. external gain) *Consider the interconnected system Σ given by* (1)*,* (2) *where each of the subsystems Σ_i has an ISS Lyapunov function V_i and the corresponding gain matrix is given by* (5)*. Assume that u enters the ISS condition via maximization, that is,*

$$\overline{\Gamma}_\mu(V_1(x_1), \dots, V_n(x_n), \|u\|) = \max\{\Gamma_\mu(V_1(x_1), \dots, V_n(x_n)), \gamma_u(\|u\|)\} ,$$

where $\gamma_u(\|u\|) = (\gamma_{1u}(\|u\|), \dots, \gamma_{nu}(\|u\|))^T$. Then, if Γ_μ is irreducible and satisfies the small gain condition

$$\Gamma_\mu(s) \not\geq s$$

the interconnected system is ISS and an ISS Lyapunov function is given by (10)*, where $\sigma \in \mathcal{K}_\infty^n$ is an arbitrary Ω-path with respect to Γ_μ and φ is a $\mathcal{K}_\infty$ function with the property*

$$\gamma_{iu} \circ \varphi(r) \leq \Gamma_{\mu,i}(\sigma(r)) , \quad i = 1, \dots, n, \tag{12}$$

where $\Gamma_{\mu,i}$ denotes the i-th row of Γ_μ.

In the next result observe that (M3) is not always necessary for the u-component of μ.

Corollary 15 (Separation in gains) *Consider the interconnected system Σ given by* (1)*,* (2) *where each of the subsystems Σ_i has an ISS Lyapunov function V_i and the corresponding gain matrix Γ is given by* (5)*. Assume that Γ is irreducible and that the gains in the ISS condition are separated, that is, there exist $\mu \in \mathrm{MAF}_n^n$, $c \in \mathbb{R}, c > 0$, and $\gamma_u \in \mathcal{K}_\infty$ such that*

$$\begin{aligned}\overline{\Gamma}_\mu(V_1(x_1), \dots, V_n(x_n), \|u\|) = \\ (c + \gamma_u(\|u\|))\ \Gamma_\mu(V_1(x_1), \dots, V_n(x_n)) .\end{aligned} \tag{13}$$

If there exists an $\alpha \in \mathcal{K}_\infty$ such that[2] *for $D = \mathrm{diag}(c \cdot id + id \cdot \alpha)$ the gain operator Γ_μ satisfies the strong small gain condition*

$$D \circ \Gamma_\mu(s) \not\geq s$$

then the interconnected system is ISS and an ISS Lyapunov function is given by (10)*, where $\sigma \in \mathcal{K}_\infty^n$ is an arbitrary Ω-path with respect to $D \circ \Gamma_\mu(s)$.*

It is worth pointing out that the work [10] contains a section showing how simply connected networks can be dealt with by resorting to results for strongly connected networks. For reasons of space, we do omit this here.

[2] The dot denotes ordinary multiplication, i.e., for $s \in \mathbb{R}_+$ we have $(c \cdot \mathrm{id})(s) = cs$ and $(\mathrm{id} \cdot \alpha)(s) = s\alpha(s)$.

V. Path construction

This section explains the relation between the small gain condition for Γ_μ and its mapping properties. Then we construct an Ω-path and prove Theorem 9 and some extensions. Let us first consider some simple particular cases to explain the main ideas. In the following subsections we then proceed to the main path construction results.

A map $T : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is *monotone* if $x \leq y$ implies $T(x) \leq T(y)$. Clearly any matrix $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ together with an aggregation $\mu \in \mathrm{MAF}_n^n$ induces a monotone map Γ_μ.

The first result applies to Γ with bounded entries, i.e., in $(\mathcal{K} \setminus \mathcal{K}_\infty) \cup \{0\}$.

Proposition 16 *Assume that $\Gamma \in (\mathcal{K} \cup \{0\})^{n\times n}$ has no zero rows and let $\mu \in \mathrm{MAF}_n^n$ be such that Γ_μ satisfies the small gain condition* (SGC)*. Assume furthermore that Γ_μ is bounded, then there exists an Ω-path with respect to Γ_μ.*

The difficulty now arises if Γ_μ happens to be unbounded, i.e., Γ contains entries of class $\mathcal{K}_\infty$.

A. *Paths for $\mathcal{K}_\infty \cup \{0\}$ gain matrices*

In this subsection we consider matrices $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$, i.e., all nonzero entries of Γ are assumed to be unbounded functions.

In this setting we assume and utilize that the graph associated to Γ is strongly connected, i.e., Γ is irreducible. So that if we consider powers $\Gamma_\mu^k(x)$, for each components i and j there exists a $k = k(i, j)$ such that $t \mapsto \Gamma_\mu^k(t \cdot e_j)_i$ is an unbounded function.

Theorem 17 *Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ be irreducible, $\mu \in \mathrm{MAF}_n^n$, and assume $\Gamma_\mu \not\geq id$. Then there exists a strictly increasing path $\sigma \in \mathcal{K}_\infty^n$ satisfying*

$$\Gamma_\mu(\sigma(r)) < \sigma(r), \quad \forall r > 0.$$

The main technical difficulty in the proof is to construct the path in the unbounded direction.

It is possible to consider the reducible case in a similar fashion. The argument is essentially an induction over the number of irreducible and zero blocks on the diagonal of the reducible operator. We cite the following result from [26, Theorem 5.10]. However, for the construction of an ISS Lyapunov function in the case of reducible Γ, we can also take different route as described [10].

Theorem 18 *Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ be reducible, $\mu \in \mathrm{MAF}_n^n$ satisfying (M4), $D = \mathrm{diag}(id + \alpha)$ for some $\rho \in \mathcal{K}_\infty$, and assume $\Gamma_\mu \circ D \not\geq id$. Then there exists a monotone and continuous operator $\tilde{D} : \mathbb{R}_+^n \to \mathbb{R}_+^n$ and a strictly increasing path $\sigma : \mathbb{R}_+ \to \mathbb{R}_+^n$ whose component functions are all unbounded, such that $\Gamma_\mu \circ \tilde{D}(\sigma) < \sigma$.*

B. General Γ_μ

In the preceding subsections we have seen that it is possible to construct Ω-paths for matrices Γ whose nonzero entries are either all bounded, or all unbounded. It remains to consider the case that the nonzero entries of Γ are partly of class $\mathcal{K}_\infty$ and partly of class $\mathcal{K} \setminus \mathcal{K}_\infty$. We can state the following result.

Proposition 19 *Let $\Gamma \in (\mathcal{K} \cup \{0\})^{n\times n}$ and let $\mu \in \mathrm{MAF}_n^n$ satisfy (M4). Assume Γ_μ satisfies* (sSGC)*. Then there exists an Ω-path for Γ_μ.*

Theorem 20 *Let $\mu = \max$ and $\Gamma \in (\mathcal{K} \cup \{0\})^{n\times n}$. If all subordinated cycles of Γ are contractions, then there exists an Ω-path with respect to Γ_μ.*

For the proof see [10].

VI. Conclusions

In this paper we have provided a method for the construction of ISS Lyapunov functions for interconnections of nonlinear ISS systems. The method applies for an interconnection of an arbitrary finite number of subsystems interconnected in an arbitrary way and satisfying a small gain condition. The small gain condition is imposed on a nonlinear gain operator Γ_μ. This operator contains the information of the topological structure of the network and the interactions between its subsystems.

An ISS Lyapunov function for such a network is given in terms of ISS Lyapunov functions of subsystems and some auxiliary functions that are constructed from nonlinear gain operator Γ_μ, provided a generalized small gain condition is satisfied.

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