A Cyclic Small-Gain Condition and an Equivalent Matrix-Like Criterion for iISS Networks

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***Abstract*— This paper considers nonlinear dynamical networks consisting of individually iISS (integral input-to-state stable) subsystems which are not necessarily ISS (input-to-state stable). Stability criteria for internal and external stability of the networks are developed in view of both necessity and sufficiency. For the sufficiency, we show how we can construct a Lyapunov function of the network explicitly under the assumption that a cyclic small-gain condition is satisfied. The cyclic small-gain condition is shown to be equivalent to a matrix-like condition. The two conditions and their equivalence precisely generalize some central ISS results in the literature. Moreover, the necessity of the matrix-like condition is established. The allowable number of non-ISS subsystems for stability of the network is discussed through several necessity conditions.**

I. Introduction

Networks we often encounter in the diverse fields of science, technology, business and management are aggregations of dynamical subsystems [23], [24]. In many cases, internal dynamics of subsystems and connecting channels are subject to various types of saturation mechanism. The notion of integral input-to-state stability (iISS) covers such a class of dynamics [26]. The notion of input-to-state stability (ISS), meanwhile, requires more stable dynamics which produces bounded state for arbitrary magnitude of input [25]. The ISS small-gain theorem establishes stability of interconnected systems if "large" nonlinear gain of one subsystem is compensated by "small" nonlinear gain of the other subsystem [15], [29]. Even if the number of subsystems is more than two, the idea still remains valid [5], [16], [18], [6], [4], [17], [30]. Due to the conservation principle underlying natural dynamics of systems, saturated energy decrease in one part is balanced by saturated energy increase in another part. Under this balance, the existence of a component dissipating energy drives the state of a system into the minimum energy level. Therefore, a large-scale system can be stable if a "more" stable subsystem which is ISS makes up for "less" stable subsystems which are not ISS. This natural observation has led to the small-gain-type criteria for iISS systems [7], [11], [1]. Very recently, the observation has been extended to networks without any restriction on network graph topology through explicit construction of a Lyapunov function [12].

The small-gain theorem for iISS networks is basically a generalization of previous developments for ISS networks. However, there remain gaps between the iISS and ISS cases, and so far no exhaustive explanation of the differences has been provided. Most of Lyapunov-based studies on ISS small-gain criteria have employed the max-type construction of a Lyapunov function V for a network as

$$V(x) = \max_i W_i(V_i(x_i)), \tag{1}$$

which is the weighted maximum of Lyapunov functions V_i of individual subsystems [14], [16], [18], [6], [4]. Weights are W_i. It has been proved in [10] that the max-type construction (1) yields a Lyapunov function only if all subsystems in the network are guaranteed to be ISS by supply rates. To deal with non-ISS subsystems, the sum-type construction

$$V(x) = \sum_i W_i(V_i(x_i)) \tag{2}$$

has been successful [7], [11]. Such construction has been explicitly shown in [12] for networks under a small-gain criterion. ISS networks are covered as a special case. However, the small-gain criterion developed there is not precisely identical with the ones developed for ISS networks. Indeed, even for ISS networks, the formulation for V_i of subsystems

$$\dot{V}_i \leq -\alpha_i(V_i(x_i)) + \sum_j \sigma_{i,j}(V_j(x_j)) \tag{3}$$

employed in [12] is different from the one

$$\dot{V}_i \leq -\alpha_i(V_i(x_i)) + \max_j \sigma_{i,j}(V_j(x_j)) \tag{4}$$

associated with the ISS result [16], [18], [17] of a cyclic small-gain condition. Can we obtain a precise extension of the cyclic small-gain condition, which has been linked with the pair of (4) and (1), to iISS networks even if we replace (1) by (2)? By establishing an affirmative answer to this question, this paper achieves the exact generalization of some central ISS results in the literature. For the maximum formulation (4) of iISS subsystems, this paper develops necessary conditions as well as an iISS small-gain criterion which is a sufficient condition for the stability of iISS networks. This paper also demonstrates that the small-gain criterion is equivalently expressed by a matrix-like condition generalizing an ISS result [6], [4]. The allowable number of non-ISS subsystems for stability of the network is discussed in terms of some necessary conditions.

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Throughout this paper, the symbols $\vee$ and $\wedge$ represent the logical sum and the logical product, respectively. The symbol $|x|$ denotes the Euclidean norm of a real vector $x \in \mathbb{R}^n$. A continuous function $\gamma : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}_+$ is said to be positive definite (written as $\gamma \in \mathcal{P}$) if it satisfies $\gamma(0) = 0$ and $\gamma(s) > 0$ holds for all $s > 0$. A continuous function γ is of class $\mathcal{K}$ (written as $\gamma \in \mathcal{K}$) if $\gamma \in \mathcal{P}$ and it is strictly increasing; it is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. We write $\gamma \in \mathcal{K} \cup \{0\}$ to indicate that γ is either of class $\mathcal{K}$ or the zero function. The symbol $\mathbf{Id}$ denotes the identity function on $\mathbb{R}_+$. For a function $\gamma \in \mathcal{P}$, we write $\gamma \in \mathcal{O}(>L)$ with a non-negative number L if there exists a positive number $K > L$ such that $\limsup_{s\to 0+} \gamma(s)/s^K < \infty$. We write $\gamma \in \mathcal{O}(L)$ when $K = L$. Let e_k for $k = 1, 2, ..., n$ be the standard basis of $\mathbb{R}^n$. Let I be an index set such that $I \subset \{1, 2, ..., n\}$. We denote by $P_I : \mathbb{R}^n \to \mathbb{R}^{\#I}$ the projection of the coordinates in $\mathbb{R}^n$ corresponding to the indices in I onto $R^{\#I}$, where $\#I$ is the cardinality of I. The anti-projection corresponding to P_I is $Q_I : \mathbb{R}^{\#I} \to \mathbb{R}^n$ defined as $x \in \mathbb{R}^{\#I} \mapsto (x_1 e_{i_1} + \ldots + x_{\#I} e_{i_{\#I}}) \in \mathbb{R}^n$, where $x = [x_1, ..., x_{\#I}]^T$ and $I = \{i_1, ... i_{\#I}\}$. For a mapping $M : \mathbb{R}^n \to \mathbb{R}^n$, we use the similar notation $M_{I,J} := P_I \circ M \circ Q_J$. For a vector $\mathbf{s} \in \mathbb{R}^n$, we write $\mathbf{s}_I := P_I(\mathbf{s})$. For vectors $a, b \in \mathbb{R}^n$ the relation $a \geq b$ is defined by $a_i \geq b_i$ for all $i = 1, \ldots, n$. The negation of $a \geq b$ is denoted by $a \not\geq b$, i.e., there exists an $i \in \{1, \ldots, n\}$ such that $a_i < b_i$. The relation $a \gg b$ is defined by $a_i > b_i$ for all $i = 1, \ldots, n$. The negation $a \not\gg b$ is the existence of an $i \in \{1, \ldots, n\}$ for which $a_i \leq b_i$ holds. Let $\overline{\mathbb{R}}_+$ denote the set of extended non-negative real numbers, i.e., $\overline{\mathbb{R}}_+ := [0, \infty]$. The mapping $M_{I,J}$ is extended to $\overline{\mathbb{R}}_+$ for $M : \overline{\mathbb{R}}^n_+ \to \overline{\mathbb{R}}^n_+$ as defined on $\mathbb{R}^n_+$ in the above. The inequalities $<$ and $\leq$ on $\mathbb{R}_+$ are extended to $\overline{\mathbb{R}}_+$ with the convention $\infty \leq \infty$. If γ is a class $\mathcal{K}_\infty$ function, its inverse γ^{-1} is of class $\mathcal{K}_\infty$. For $\gamma \in \mathcal{K} \backslash \mathcal{K}_\infty$, its inverse γ^{-1} is defined on the finite interval $[0, \lim_{\tau\to\infty} \gamma(\tau))$ since the continuous function γ is strictly increasing and $\gamma(0) = 0$. For $\gamma \in \mathcal{K}$, an operator $\gamma^\ominus$: $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is defined as $\gamma^\ominus(s) := \sup\{v \in \mathbb{R}_+ : s \geq \gamma(v)\}$. That is, we have $\gamma^\ominus(s) = \infty$ for $s \geq \lim_{\tau\to\infty} \gamma(\tau)$, and $\gamma^\ominus(s) = \gamma^{-1}(s)$ elsewhere. For non-decreasing $\omega \in \mathcal{P}$, its extension ω: $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is defined as $\omega(s) := \sup_{v \in \{w \in \mathbb{R}_+ : w \leq s\}} \omega(v)$. Using these conventions for ω, $\gamma \in \mathcal{K}$, we have $\omega \circ \gamma^\ominus(s) = \lim_{\tau\to\infty} \omega(\tau)$ for $s \geq \lim_{\tau\to\infty} \gamma(\tau)$. The identity $\gamma^\ominus = \gamma^{-1} \in \mathcal{K}$ holds if and only if $\gamma \in \mathcal{K}_\infty$. It is important that, in the case of $\gamma \in \mathcal{K} \backslash \mathcal{K}_\infty$, we have only $\gamma \circ \gamma^\ominus(s) \leq s$ for $s \in \overline{\mathbb{R}}_+$ although $\gamma^\ominus \circ \gamma(s) = s$ for $s \in \overline{\mathbb{R}}_+$. For $\gamma, \omega \in \mathcal{K} \cup \{\zeta^\ominus : \zeta \in \mathcal{K}\}$, $\gamma \circ \omega(s) \leq s$, $\forall s \in \mathbb{R}_+$ is equivalent to $\omega \circ \gamma(s) \leq s$, $\forall s \in \mathbb{R}_+$. This equivalence also holds for $\gamma \in \mathcal{K} \cup \{\zeta^\ominus : \zeta \in \mathcal{K}\}$ and a non-decreasing function $\omega \in \mathcal{P}$. Due to space limitation, all proofs are omitted.

II. Network of iISS Systems

Consider the dynamical network Σ described by

$$\Sigma : \ \dot{x} = f(x, r), \tag{5}$$

where $x(t) \in \mathbb{R}^N$ is the state vector of Σ, and $r(t) \in \mathbb{R}^K$ is the external input which is assumed to be measurable, locally essentially bounded. Assume that the function $f : \mathbb{R}^{N+K} \to \mathbb{R}^N$ is locally Lipschitz. Let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

$$x_i \in \mathbb{R}^{N_i}, \ N = \sum_{i=1}^{n} N_i, \ r_i \in \mathbb{R}^{K_i}, \ K = \sum_{i=1}^{n} K_i$$

and the network Σ consists of subsystems Σ_i governed by

$$\Sigma_i : \ \dot{x}_i = f_i(x_1, \ldots, x_n, r_i), \quad i = 1, 2, ..., n. \tag{6}$$

The integer $n \geq 2$ is the number of subsystems. Instead of requiring the precise knowledge of f_i, this paper assumes that a dissipation inequality of each subsystem Σ_i is known.

Assumption 1: For each $i = 1, 2, ..., n$, there exist a positive definite, radially unbounded $\mathbf{C}^1$ function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ and continuous functions $\alpha_i \in \mathcal{K}$, $\sigma_{i,j}, \kappa_i \in \mathcal{K} \cup \{0\}$ such that

$$\frac{\partial V_i}{\partial x_i} f_i \leq -\alpha_i(V_i(x_i)) + \max\left\{ \max_{j \in \{1,2,...,n\}} \sigma_{i,j}(V_j(x_j)),\ \kappa_i(|r_i|) \right\} \tag{7}$$

holds for all $x_j \in \mathbb{R}^{N_j}$ and $r_j \in \mathbb{R}^{K_j}$, $j = 1, 2, ..., n$, where $\sigma_{i,i} \equiv 0$, $i = 1, 2, ..., n$.

Note that $\dot{V}_i = \frac{\partial V_i}{\partial x_i} f_i$ is the time derivative of V_i along $x_i(t)$. We consider α_i, $\sigma_{i,j}$, κ_i instead of f_i. The inequality (7) is called a dissipation inequality specifying that each subsystem Σ_i is integral input-to-state stable (iISS) with respect to the inputs x_j, $j \neq i$ and r. The function V_i is an iISS Lyapunov function for Σ_i considered separately [2]. Note that a subsystem Σ_i prescribed by (7) is guaranteed to be input-to-state stable (ISS) with respect to the inputs x_j, $j \neq i$ and r if and only if Assumption 1 can be satisfied by a pair (V_i, α_i) with $\alpha_i \in K_\infty$ [28], [2], [27]. If V_i is fixed, the requirement corresponding to $\alpha_i \in \mathcal{K}_\infty$ for Σ_i to be ISS can be relaxed into $\lim_{s\to\infty} \alpha_i(s) = \infty \vee \lim_{s\to\infty} \alpha_i(s) \geq \lim_{s\to\infty} \max\{\max_{j \in \{1,2,...,n\}} \sigma_{i,j}(s), \kappa_i(s)\}$. It is important to recall that the set of ISS systems is a strict subset of the set of iISS systems [26]. This paper employs the dissipation inequality (7) in which $\max_j$ defines the interaction of subsystems. The interaction could be stated using $\sum_j$ (i.e., sum over j) instead of $\max_j$ in (7). The formulation of type (7) is called maximization aggregation in [6], while the $\sum_j$ formulation is called summation aggregation. The summation aggregation has been investigated by some preceeding results [12], [9], [3], [10], [22]. Defining the supply rates of subsystems in the maximization form, the goal of this paper is to construct an iISS Lyapunov function $V(x)$ of the network Σ with respect to input r and state x, and to find a condition under which such construction is possible. In this paper the network Σ is said to be 0-GAS if the origin $x = 0$ is globally asymptotically stable for $r(t) \equiv 0$.

Remark 1: Using [28], [2], [27], we can verify that a subsystem Σ_i prescribed by (7) is guaranteed to be ISS if and only if there exist β_i, $\chi_{i,j}$, $\chi_i \in \mathcal{K}$ ($\chi_{i,i} = 0$) and a $\mathbf{C}^1$

function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ satisfying the implication

$$|x_i| \geq \max\left\{\max_j \chi_{i,j}(|x_j|),\ \chi_i(|r|)\right\} \ \Rightarrow\ \dot{V}_i \leq -\beta_i(|x_i|). \tag{8}$$

The characterization (8) for subsystems, referred to as the implication formulation (or the gain margin formulation), is used for ISS networks in [16], [18], [6], [4]. General iISS subsystems reject such an implication formulation [2].

Remark 2: The developments in this paper remain valid even for $\sigma_{i,i} \neq 0$ in (7). It only introduces diagonal elements [21] in $S(\mathbf{s})$, i.e., loops, which can be considered as cycles of length one, to the network graph in the next sections. It amounts to requiring the internal gain to be less than $\mathbf{Id}$. This paper employs $\sigma_{i,i} = 0$ so that (7) by itself retains ISS or iISS of subsystems. For broadened classes of systems such as retarded systems, $\sigma_{i,i} \neq 0$ is useful [17], [13].

III. A Characterization by Sum-Type Lyapunov Functions

Define A, S, D, Λ: $\mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$A(\mathbf{s}) = \begin{bmatrix} \alpha_1(s_1) \\ \alpha_2(s_2) \\ \vdots \\ \alpha_n(s_n) \end{bmatrix}, \quad S(\mathbf{s}) = \begin{bmatrix} \max_j \sigma_{1,j}(s_j) \\ \max_j \sigma_{2,j}(s_j) \\ \vdots \\ \max_j \sigma_{n,j}(s_j) \end{bmatrix}$$

$$D(\mathbf{s}) = \begin{bmatrix} s_1 + \delta_1(s_1) \\ s_2 + \delta_2(s_2) \\ \vdots \\ s_n + \delta_n(s_n) \end{bmatrix}, \quad \Lambda(\mathbf{s}) = \begin{bmatrix} \lambda_1(s_1) \\ \lambda_2(s_2) \\ \vdots \\ \lambda_n(s_n) \end{bmatrix},$$

where $\mathbf{s} = [s_1, s_2, ..., s_n]^T \in \mathbb{R}_+^n$, and λ_i, δ_i are auxiliary functions to be specified below. The operators A, D and Λ have the same diagonal structure while S is not diagonal. Appropriate functions λ_i, δ_i lead us to a Lyapunov function of the network Σ. The following demonstrates this fact.

Theorem 1: Suppose that there exist continuous functions $\lambda_i, \delta_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, ..., n$, such that

$$\lambda_i(s) > 0, \quad \forall s \in (0, \infty), \qquad i = 1, 2, ..., n \tag{9}$$

$$\int_1^\infty \lambda_i(s) ds = \infty, \qquad i = 1, 2, ..., n \tag{10}$$

$$\left\{\{\alpha_i \in \mathcal{K}_\infty \ \wedge\ \delta_i \in \mathcal{K}_\infty\} \ \vee\ \limsup_{s\to\infty} \lambda_i(s) < \infty\right\}, \quad i = 1, 2, ..., n \tag{11}$$

$$\mathbf{Id} + \delta_i \in \mathcal{K}_\infty, \qquad i = 1, 2, ..., n \tag{12}$$

$$\delta_i(s) > 0,\ \forall s \in (0, \lim_{\tau\to\infty} \alpha_i(\tau)), \quad i = 1, 2, ..., n \tag{13}$$

$$\Lambda(\mathbf{s})^T[-D^{-1} \circ A(\mathbf{s}) + S(\mathbf{s})] \leq 0, \quad \forall \mathbf{s} \in \mathbb{R}_+^n. \tag{14}$$

Then Σ is iISS with respect to input r and state x. If

$$\alpha_i \in \mathcal{K}_\infty, \qquad i = 1, 2, ..., n \tag{15}$$

$$\liminf_{s\to\infty} \lambda_i(s) > 0, \quad i = 1, 2, ..., n \tag{16}$$

are satisfied additionally, then the network Σ is ISS. Furthermore, an iISS (ISS) Lyapunov function is

$$V(x) = \sum_{i=1}^n \int_0^{V_i(x_i)} \lambda_i(s) ds\ . \tag{17}$$

Theorem 1 for the maximization supply rate (7) is parallel to the result developed in [10] for the summation supply rate (3). The difference between them appears in the definition of S and the constraint on δ_i. This paper relaxes $\delta_i \in \mathcal{K}_\infty$ required in [10] into (11). The formula (17) of Lyapunov functions is referred to as the sum-type construction, which was used for the summation supply rate in [10], [12]. Since the maximization and summation supply rates result in different S, the operators Λ achieving (14) are different from each other. As in the summation formulation of supply rates [12], it will be shown in Theorem 5 for the maximization formulation of supply rates that the sum-type construction of a Lyapunov function (17) is crucial in establishing stability of the networks involving non-ISS subsystems.

IV. An Explicit Lyapunov Function

This section explicitly constructs Λ which yields an iISS Lyapunov function of the network Σ in Theorem 1. Let the direct graph G be associated with the network Σ with the vertex set $\mathcal{V}(G)$ and the arc set $\mathcal{A}(G)$. The vertices are subsystems, $i = 1, 2, ..., n$. The pair (i, j) is an element of the arc set $\mathcal{A}(G)$ if and only if $\sigma_{i,j} \neq 0$. An arc (i, j) is directed away from the j-th vertex and directed toward the i-th vertex. Define the weight of the arc (i, j) of G as the function $\sigma_{i,j}(s)$. The weight of the vertex i is defined as $\alpha_i^\ominus$: $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$. As usual, a walk is an alternating sequence of vertices and connecting arcs, beginning and ending with a vertex. A walk is a path if it has no repeated vertices. A walk is a cycle if it starts and ends at the same vertex but otherwise has no repeated vertices. Given a path or a cycle U of length k, we write $|U| = k$ and $U = (u(1), u(2), ..., u(k), u(k+1))$, where $u(i)$'s are "all" vertices of U listed in the "reversed" order of appearance. The starting vertex of the path U is $u(k + 1)$, and the ending vertex is $u(1)$. If U is a cycle, we have $u(1) = u(k+1)$. Let $\mathcal{C}(G)$ denote the set of all directed cycle graphs contained in the graph G. We define $F_{i,j} \in \mathcal{K}$ for $i, j = 1, 2, ..., n$ as follows:

Lemma 1: Consider $\alpha_i \in \mathcal{K}$ and $\sigma_{i,j} \in \mathcal{K} \cup \{0\}$, $\sigma_{i,i} = 0$, $i, j = 1, 2, ..., n$, satisfying

$$\left\{\lim_{s\to\infty} \alpha_j(s) = \infty \ \vee\ \lim_{s\to\infty} \max_{i\in\{1,2,...,n\}} \sigma_{i,j}(s) < \infty\right\}, \quad j = 1, 2, ..., n. \tag{18}$$

Suppose that there exist $c_i > 1$, $i = 1, 2, ..., n$ such that

$$\alpha_{u(1)}^\ominus \circ c_{u(1)}\sigma_{u(1),u(2)} \circ \alpha_{u(2)}^\ominus \circ c_{u(2)}\sigma_{u(2),u(3)} \circ \cdots \circ \alpha_{u(|U|)}^\ominus \circ c_{u(|U|)}\sigma_{u(|U|),u(|U|+1)}(s) \leq s,\ \forall s \in \mathbb{R}_+ \tag{19}$$

holds for all cycles $U \in \mathcal{C}(G)$. Let τ_i be such that

$$1 < \tau_i < c_i, \quad i = 1, 2, ..., n. \tag{20}$$

Then there exist $F_{i,j} \in \mathcal{K}$, $i, j = 1, 2, ..., n$, satisfying

$$\alpha_i^\ominus \circ c_i F_{i,i}(s) \leq s, \qquad \forall s \in \mathbb{R}_+, \qquad i = 1, 2, ..., n \tag{21}$$

$$F_{i,j}(s) \geq \max\left\{ \max_{\substack{1 \leq q \leq n \\ q \neq i, q \neq j}} F_{i,q} \circ \alpha_q^\ominus \circ \tau_q F_{q,j}(s),\ \sigma_{i,j}(s)\right\},\ \forall s \in \mathbb{R}_+,\ i, j = 1, 2, ..., n \tag{22}$$

$$\lim_{s\to\infty} F_{i,j}(s) < \infty \vee \lim_{s\to\infty} \max\left\{ \max_{\substack{1\leq q\leq n \\ q\neq i, q\neq j}} F_{i,q}\circ\alpha_q^{\ominus}\circ\tau_q F_{q,j}(s), \sigma_{i,j}(s), \right\} = \infty,\ i,j=1,2,...,n \quad (23)$$

$$\{\lim_{s\to\infty}\alpha_j(s)=\infty \vee \lim_{s\to\infty}\max_{i\in\{1,2,...,n\}} F_{i,j}(s)<\infty\},\quad j=1,2,...,n. \quad (24)$$

The function $F_{i,j} \in \mathcal{K}$ is essentially the maximum of nonlinear gain functions defined as the composite mappings of the alternate sequences of arc and vertex weights along all walks from j to i. The condition (19) guarantees that we do not have to evaluate walks which are neither paths nor cycles. If the maximum nonlinear gain function is not strictly increasing, it is replaced by a strictly increasing function computed with the help of (21). The following theorem shows an explicit formula for computing Λ.

Theorem 2: Consider $\alpha_i \in \mathcal{K}$ and $\sigma_{i,j} \in \mathcal{K}\cup\{0\}$, $\sigma_{i,i} = 0$, $i,j = 1,2,...,n$, satisfying (18). Suppose that there exist $c_i > 1$, $i = 1,2,...,n$ such that (19) holds for all cycles $U\in\mathcal{C}(G)$. Let τ_i and $\psi \geq 0$ be such that (20) and

$$(\tau_i/c_i)^{\psi} \leq \tau_i - 1, \quad i=1,2,...,n \quad (25)$$

are satisfied. Pick class $\mathcal{K}$ functions $F_{i,j}$, $i,j = 1,2,...,n$, such that (21)-(24) are satisfied (guaranteed by Lemma 1). Define class $\mathcal{K}$ functions $\overline{\lambda}_i$, $i = 1,2,...,n$, by

$$\overline{\lambda}_i(s) = \left[\frac{1}{\tau_i}\alpha_i(s)\right]^{\psi} \prod_{j\in\mathcal{V}(G)-\{i\}} [F_{j,i}(s)]^{\psi+1}. \quad (26)$$

Let ν_i: $(0,\infty) \to \mathbb{R}_+$, $i = 1,2,...,n$, be continuous functions fulfilling

$$0 < \nu_i(s) < \infty, \quad s\in(0,\infty) \quad (27)$$

$$\lim_{s\to\infty}\alpha_i(s) = \infty \vee \lim_{s\to\infty}\nu_i(s) < \infty \quad (28)$$

$$\overline{\lambda}_i(s)\nu_i(s) : \text{non-decreasing continuous for } s\in(0,\infty) \quad (29)$$

$$\nu_{u(j)}\circ\alpha_{u(j)}^{\ominus}\circ\tau_{u(j)}\sigma_{u(j),u(j+1)}(s) \leq (c_{u(j+1)}/\tau_{u(j+1)})^{\psi}(\tau_{u(j+1)}-1)\nu_{u(j+1)}(s), \quad \forall s\in(0,\infty),\ j=1,2,...,|U| \quad (30)$$

for all cycles $U \in \mathcal{C}(G)$. Then the non-decreasing continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,2,...,n$, defined by

$$\lambda_i(s) = \overline{\lambda}_i(s)\nu_i(s), \quad s\in(0,\infty), \quad i=1,2,...,n \quad (31)$$

$$\lambda_i(0) = \lim_{s\to 0^+}\overline{\lambda}_i(s)\nu_i(s) \quad (32)$$

achieve (9)-(14) with $\delta_i(s) = b_i s$, $i = 1,2,...,n$, for some $b_i > 0$. Moreover, the property (16) holds if (15).

By virtue of Theorem 1, the function V in (17) with (31) is an iISS Lyapunov function of the network Σ with input r and state x. The collective condition (19) is a sufficient condition for the iISS of Σ. Theorem 2 allows some subsystems to be non-ISS. It realizes the intuitive idea of compensating vulnerable subsystems with constraining subsystems in feedback for the general network topology. Clearly, the constants τ_i, ψ fulfilling (20) and (25) always exist. The functions ν_i, $i = 1,2,...,n$, achieving (27)-(30) also always exist. Indeed, a simple choice is $\nu_1(s) = ... = \nu_n(s) = \text{constant} > 0$. We are able to replace the linear functions $c_i s$ in (19) with nonlinear functions $s + \delta_i(s)$ at the expense of some technical complexity in the formula for λ_i following the idea in [11].

Remark 3: When $\alpha_i \in \mathcal{K}_\infty$, $i = 1,2,...,n$, i.e., all subsystem are ISS, Theorem 1 ensures that V constructed as in (17) with (31) is an ISS Lyapunov function of Σ. For networks consisting of ISS subsystems, the condition (19) is exactly the equivalent variant of the cyclic small-gain condition developed for ISS networks in [16], [18], [6]. The maximization supply rate (7) allows us to attain this precise correspondence. It is stressed that the stability condition presented in [12] for the summation supply rates (3) is not precisely the same as the cyclic small-gain condition (19) (i.e., the one in [16], [18], [6]) even if all subsystem are ISS. Compared with the result in [12], the small-gain condition (19) directly uses α_i and $\sigma_{i,j}$ appearing in the supply rates of subsystems (7). In this paper, neither the stability criterion (19) nor the construction of the Lyapunov function V requires the process of covering the network graph by subgraphs on which the summation result is based.

V. Matrix-Like Small-Gain Conditions

For networks consisting of ISS subsystems, the studies in [20], [17] demonstrated that the cyclic small-gain condition (19) is equivalent to a matrix-like condition. The equivalence was proved only for the maximization aggregation of $\mathcal{K}_\infty$ nonlinear gains. In the iISS formulation this paper employs, the network is allowed to have multiple non-ISS subsystems which lead to the small-gain condition (19) containing several $\alpha_i^{\ominus}$ which are of neither $\mathcal{K}_\infty$ nor $\mathcal{K}$ and involve $\overline{\mathbb{R}}_+$. This section generalizes the equivalence to networks involving non-ISS subsystems making use of the maximization supply rate (7). Define $A^{\ominus} : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+^n$ by

$$A^{\ominus}(\mathbf{s}) = [\alpha_1^{\ominus}(s_1), \alpha_2^{\ominus}(s_2), \cdots, \alpha_n^{\ominus}(s_n)]^T, \qquad \mathbf{s}\in\mathbb{R}_+^n.$$

For $\alpha_i \in \mathcal{K}_\infty$, $i = 1,2,...,n$, we have $A\circ A^{\ominus}(\mathbf{s}) = \mathbf{s}$ for all $\mathbf{s}\in\mathbb{R}_+^n$. Admitting $\alpha_i \in \mathcal{K}\setminus\mathcal{K}_\infty$ implies $A\circ A^{\ominus} \neq \mathbf{Id}$ on $\overline{\mathbb{R}}_+^n$ although $A^{\ominus}\circ A = \mathbf{Id}$ holds on $\overline{\mathbb{R}}_+^n$. Nevertheless, we have the following for matrix-like conditions.

Lemma 2: Suppose that $\alpha_i \in \mathcal{K}$, $\sigma_{i,j} \in \mathcal{K}\cup\{0\}$ and $\mathbf{Id}+\delta_i \in \mathcal{K}_\infty$ for $i,j = 1,2,...,n$. Then the following three properties are equivalent to one another:

$$A^{\ominus}\circ D\circ S(\mathbf{s}) \not\geq \mathbf{s}, \quad \forall\mathbf{s}\in\mathbb{R}_+^n\setminus\{0\} \quad (33)$$

$$D\circ S(\mathbf{s}) \not\geq A(\mathbf{s}), \quad \forall\mathbf{s}\in\mathbb{R}_+^n\setminus\{0\} \quad (34)$$

$$S(\mathbf{s}) \not\geq D^{-1}\circ A(\mathbf{s}), \quad \forall\mathbf{s}\in\mathbb{R}_+^n\setminus\{0\}. \quad (35)$$

The above property allows us to verify the equivalence we are pursuing for iISS networks.

Proposition 1: Consider $\alpha_i \in \mathcal{K}$, $\sigma_{i,j} \in \mathcal{K}\cup\{0\}$, $\sigma_{i,i} = 0$ and $\delta_i \in \mathcal{K}_\infty$ for $i,j = 1,2,...,n$. Then the inequality

$$\alpha_{u(1)}^{\ominus}\circ(\mathbf{Id}+\delta_{u(1)})\circ\sigma_{u(1),u(2)}\circ \alpha_{u(2)}^{\ominus}\circ(\mathbf{Id}+\delta_{u(2)})\circ\sigma_{u(2),u(3)}\circ\cdots\circ \alpha_{u(|U|)}^{\ominus}\circ(\mathbf{Id}+\delta_{u(|U|)})\circ\sigma_{u(|U|),u(|U|+1)}(s) < s, \quad \forall s\in\mathbb{R}_+\setminus\{0\} \quad (36)$$

holds for all cycles $U \in \mathcal{C}(G)$ if and only if (34) is satisfied.

Notice that there exist $\delta_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$, satisfying (36) for all cycles $U \in \mathcal{C}(G)$ if and only if there exist (possibly different) $\delta_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$, such that

$$\alpha^{\ominus}_{u(1)} \circ (\mathbf{Id} + \delta_{u(1)}) \circ \sigma_{u(1),u(2)} \circ \alpha^{\ominus}_{u(2)} \circ (\mathbf{Id} + \delta_{u(2)}) \circ \sigma_{u(2),u(3)} \circ \cdots \circ \alpha^{\ominus}_{u(|U|)} \circ (\mathbf{Id} + \delta_{u(|U|)}) \circ \sigma_{u(|U|),u(|U|+1)}(s) \le s, \quad \forall s \in \mathbb{R}_+ \tag{37}$$

is satisfied for all cycles $U \in \mathcal{C}(G)$. Indeed, a strict inequality $<$ only on $(0, \infty)$ is obtained from $\le$ by replacing $\delta_i(s)$ with $\delta_i(s)/2$. The converse is trivial.

VI. Necessity Criteria

This section develops necessary conditions for stability properties of the network given in the maximization formulation of supply rates (7). The developments not only confirm that the necessary conditions proved previously for the summation supply rates [8], [9] can be rewritten for the maximization supply rates, but also highlight a fundamental difference. For investigating necessary conditions for the stability, this section considers "sets" of networks defined by dissipation inequalities of subsystems without looking at particular "elements" defined by differential equations.

Definition 1: Given $\alpha_i \in \mathcal{K}$, $\sigma_{i,j}, \kappa \in \mathcal{K} \cup \{0\}$, $\sigma_{i,i} = 0$, and positive integers n, N_i, K_i for $i, j = 1, 2, ..., n$, let $\mathcal{S}(n, N_*, K_*, \alpha_*, \sigma_{*,*}, \kappa_*)$ denote the set of networks Σ consisting of subsystems Σ_i, $i = 1, 2, ..., n$, in the form of

$$\dot{x}_i = f_i(x_1, \ldots, x_n, r_i), \quad x_i \in \mathbb{R}^{N_i}, \; r_i \in \mathbb{R}^{K_i} \tag{38}$$

$$f_i(0, ..., 0, 0) = 0, \quad f_i \text{ is locally Lipschitz} \tag{39}$$

for which there exist positive definite and radially unbounded $\mathbf{C}^1$ functions V_i: $\mathbb{R}^{N_i} \to \mathbb{R}$ such that (7) holds for all $x_j \in \mathbb{R}^{N_j}$ and $r_j \in \mathbb{R}^{K_j}$, $j = 1, 2, ..., n$.

The Lipschitzness imposed on f_i is only for guaranteeing the existence of a unique maximal solution of the network Σ. For brevity, we write $\mathcal{S}$ instead of $\mathcal{S}(n, N_*, K_*, \alpha_*, \sigma_{*,*}, \kappa_*)$. All the developments in this section hold true even if N_1, ..., N_n and K_1, ... , K_n are dropped from the definition of $\mathcal{S}$. Allowing N_i and K_i to be prescribed makes derived necessary conditions better, i.e., the necessity remains true with respect to such narrowly specified sets. Throughout this section, we assume the following:

Assumption 2: The functions α_i, $\sigma_{i,j}$ and κ_i are continuously differentiable on $(0, \infty)$ and satisfy $\alpha_i \in \mathcal{O}(1)$ and $\sigma_{i,j}, \kappa_i \in \mathcal{O}(>0)$ for $i, j = 1, 2, ..., n$, $j \ne i$.

Define a mapping $M_0 : \mathbb{R}^n_+ \to \mathbb{R}^n$ by

$$M_0(\mathbf{s}) := -A(\mathbf{s}) + S(\mathbf{s}). \tag{40}$$

Then the following can be proved.

Theorem 3: If the network Σ is 0-GAS for all $\Sigma \in \mathcal{S}$, then

$$M_0(\mathbf{s}) \not\ge 0, \quad \forall \mathbf{s} \in \mathbb{R}^n_+ \setminus \{0\}. \tag{41}$$

The necessary condition (41) in the maximum supply rate formulation (7) is a counterpart of the topological separation condition given in [22], [9] for the summation supply rates. In contrast to the result [19], [22] based on comparison systems, this paper establishes the necessity condition (41) for networks defined on the original space of x of a specified dimension. By evaluating the limit of (41) toward ∞, $\lim_{\tau\to\infty} S(\mathbf{s})|_{s_1=\ldots=s_n=\tau} \not\gg \lim_{\tau\to\infty} A(\mathbf{s})|_{s_1=\ldots=s_n=\tau}$ is obtained. Thus, we have the following:

Corollary 1: If the network Σ is 0-GAS for all $\Sigma \in \mathcal{S}$, there exists an integer $i \in \{1, 2, ..., n\}$ such that

$$\lim_{s\to\infty} \alpha_i(s) \ge \lim_{s\to\infty} \max_{j\in\{1,2,...,n\}} \sigma_{i,j}(s). \tag{42}$$

It is worth noting that the property (42) holds if and only if a subsystem prescribed by (7) is guaranteed to be ISS with respect to input x_j $(j \ne i)$ for $r_i(t) \equiv 0$ [28], [2].

The condition (41) is necessary for the 0-GAS even if the network is decomposed into several blocks. In fact, the condition (41) is satisfied only if

$$[M_0]_{\mathcal{V}(U),\mathcal{V}(U)}(\hat{\mathbf{s}}) \not\ge 0, \quad \forall \hat{\mathbf{s}} \in \mathbb{R}^{\#U}_+ \setminus \{0\} \tag{43}$$

holds for any induced subgraph U of G. Here, $\#U$ denotes the order of the induced subgraph U , i.e., $\#U = \#\mathcal{V}(U)$. The property (43) yields two corollaries.

Corollary 2: Suppose that the network Σ is 0-GAS for all $\Sigma \in \mathcal{S}$. If there exists an integer $i \in \{1, 2, ..., n\}$ such that

$$\lim_{s\to\infty} \alpha_i(s) < \lim_{s\to\infty} \min_{k\in\{1,2,...,n\}\setminus\{i\}} \sigma_{ik}(s) \tag{44}$$

is satisfied, then

$$\lim_{s\to\infty} \alpha_j(s) \ge \lim_{s\to\infty} \min_{k\in\{1,2,...,n\}\setminus\{j\}} \sigma_{jk}(s) \tag{45}$$

holds for each $j \in \{1, 2, ..., n\} \setminus \{i\}$.

Therefore, the number of subsystems which are non-ISS with respect to *input from every single subsystem* cannot be more than one. It is, however, emphasized that a subsystem is ISS with respect to the null input from a disconnected subsystem. Thus, (45) holds if there exists a subsystem Σ_k which does not feed x_k into Σ_j, i.e., $\sigma_{jk} = 0$.

Corollary 3: Suppose that the network Σ is 0-GAS for all $\Sigma \in \mathcal{S}$. If the directed graph associated with $\mathcal{S}$ is complete, there exists at most a single $i \in \{1, 2, ..., n\}$ such that

$$\lim_{s\to\infty} \alpha_i(s) < \lim_{s\to\infty} \sigma_{ij}(s), \; j \in \{1, 2, ..., n\} \setminus \{i\}. \tag{46}$$

For sets $\mathcal{S}$ which do not form complete graphs, the network Σ can be 0-GAS for all $\Sigma \in \mathcal{S}$ even if the number of subsystems which are not ISS with respect to each coupling channel can be more than one. Such an example is a cycle network given in Remark 2 of [8].

We next discuss stability with respect to external signals.

Theorem 4: If the network Σ is ISS with respect to input r for all $\Sigma \in \mathcal{S}$, then

$$\lim_{s\to\infty} \alpha_i(s) \ge \lim_{s\to\infty} \kappa_i(s), \quad i = 1, 2, ..., n. \tag{47}$$

This theorem shows that the decay rate of all subsystems is necessarily larger than the influence of exogenous signals for ISS of the overall network. For example, $\kappa_1, ..., \kappa_n \in \mathcal{K}_\infty$ requires $\alpha_1, ..., \alpha_n \in \mathcal{K}_\infty$ for the ISS of Σ. Although the result in [9] corresponding to Theorems 4 is slightly weaker than (47), the necessity of (47) can be also verified in the summation case.

Now, we try to relate iISS and ISS of the network to the matrix-like conditions in Section V in view of necessity. Define the operator $M : \mathbb{R}_+^n \to \mathbb{R}^n$ by

$$M(\mathbf{s}) := -D^{-1} \circ A(\mathbf{s}) + S(\mathbf{s}). \tag{48}$$

By virtue of Lemma 2, Theorem 3 guarantees the existence of continuous functions δ_1, δ_2, ..., $\delta_n : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mathbf{Id} + \delta_i \in \mathcal{K}_\infty, \quad i = 1, 2, ..., n \tag{49}$$

$$M(\mathbf{s}) \not\geq 0, \quad \forall \mathbf{s} \in \mathbb{R}_+^n \setminus \{0\}. \tag{50}$$

The functions δ_i in (50) describe the gap between zero and the minimum row of M_0. In the summation formulation of supply rates (3), we can verify the necessity of a "gap" function $\delta_i \in \mathcal{K}_\infty$ for securing ISS for all $\Sigma \in \mathcal{S}$ [9]. It, however, cannot be established in the the maximization formulation (7). The next proposition, which is a sufficient condition, demonstrates this impossibility.

Proposition 2: Let $n = 2$. Consider α_1, α_2, $\sigma_{1,2}$, $\sigma_{2,1}$, κ_1, $\kappa_2 \in \mathcal{K}_\infty$. If (41) holds, the network Σ is ISS with respect to input r for all $\Sigma \in \mathcal{S}$.

For $i = 1, 2$, consider $\mu_i \in \mathcal{P}$ satisfying $\mathbf{Id} + \mu_i \in \mathcal{K}_\infty$ and $\limsup_{s\to\infty} \mu_i(s) = 0$. Then the functions $\alpha_i = (\mathbf{Id} + \mu_i) \circ \sigma_{3-i,i}$, $i = 1, 2$, fulfill (41). However, the condition (50) is not satisfied if a function δ_i is of class $\mathcal{K}_\infty$. Therefore, Proposition 2 shows that the existence of a function $\delta_i \in \mathcal{K}_\infty$ is not always necessary for guaranteeing ISS for all $\Sigma \in \mathcal{S}$ defined with (7), which contrasts sharply with the case of summation supply rates (3) discussed in [9]. Recall that iISS is weaker than ISS and stronger than 0-GAS. The above example suggests that necessary conditions for iISS which are more specific than (41) can be pursued only on a case by case basis in the maximization supply rates.

It can be proved that (18) is necessary for constructing a sum-type iISS Lyapunov function of the network Σ unless we restrict the influence of disturbances r.

Proposition 3: Suppose that there exist continuously differentiable $W_i \in \mathcal{K}_\infty$, $i = 1, 2..., n$, such that V defined by (2) is an iISS Lyapunov function with respect to input r and state x for all $\Sigma \in \mathcal{S}$. If $\lim_{s\to\infty} \alpha_i(s) \leq \lim_{s\to\infty} \kappa_i(s)$ is satisfied for $i = 1, 2, ..., n$, then the property (18) holds.

The constraint (18) can be replaced by a milder condition if we do not consider stability of Σ with respect to the external signal r (see [1], [11] for $n = 2$).

Finally, we demonstrate the advantage of using the sum-type Lyapunov function (2) over the max-type one (1).

Theorem 5: For a function V in the form of (1), let $V^\circ(x; \dot{x})$ denote the Clarke generalized derivative at x in the direction of $\dot{x}$. If there exist continuously differentiable $W_i \in \mathcal{K}_\infty$, $i = 1, 2..., n$, such that all $\Sigma \in \mathcal{S}$ satisfy $V^\circ(x; \dot{x}) \leq 0$ for all $x \in \mathbb{R}^N$ with $r(t) \equiv 0$, then

$$\lim_{s\to\infty} \alpha_i(s) \geq \lim_{s\to\infty} \max_{j\in\{1,2,...,n\}} \sigma_{i,j}(s), \quad i = 1, 2, ..., n. \tag{51}$$

Property (51) in Theorem 5 implies that the max-type Lyapunov function (1) is capable of establishing stability of the network Σ only if all subsystems fulfilling $\Sigma \in \mathcal{S}$ are guaranteed to be ISS. In contrast, as demonstrated in Section IV, the sum-type Lyapunov function (2) can ensure the stability of the network Σ involving non-ISS subsystems.

Remark 4: Since we allow subsystems to be non-ISS, the condition (41) cannot guarantee 0-GAS of the network without additional assumptions. There are pathological non-ISS subsystems for which the no-gap small-gain condition (41) does not imply the 0-GAS [1], [11]. Proposition 2 fails unless the involved functions are assumed to be unbounded.

VII. A Key to Necessity Results

The necessity criteria in Section VI are based on the existence of subsystems (6) that perfectly fit given supply rates in the maximization formulation (7). The existence of such subsystems were proved in the summation formulation (3) of supply rates in [8]. An important point there was that the subsystems admitting unique trajectories were explicitly constructed on the original state space $x_i \in \mathbb{R}^{N_i}$ with arbitrarily "specified" dimension $N_i > 0$. See Definition 1 followed by several remarks. The next lemma extends the result to the formulation (7) of supply rates.

Lemma 3: Suppose that functions $\alpha_i \in \mathcal{P}$, $\sigma_{ij} \in \mathcal{K} \cup \{0\}$, $\kappa_i \in \mathcal{K} \cup \{0\}$, $i, j = 1, 2, ...n$, real numbers $\delta \geq 0$, $\bar{\epsilon}_i > 0$ and integers $N_i > 0$, $K_i > 0$,, $n > 0$ are given. Assume that α_i, σ_{ij} and κ_i are of class $\mathbf{C}^1$ on $(0, \infty)$ and satisfy $\alpha_i \in \mathcal{O}(1)$ and $\sigma_{ij}, \kappa_i \in \mathcal{O}(>0)$ for $i, j = 1, 2, ...n$. Let $N := \sum_{i=1}^n N_i$. Then there exist locally Lipschitz functions f_i: $\mathbb{R}^{N+K_i} \to \mathbb{R}^{N_i}$, positive definite and radially unbounded $\mathbf{C}^1$ functions V_i: $\mathbb{R}^{N_i} \to \mathbb{R}$, and a real number $\epsilon_i \in [0, \bar{\epsilon}_i]$, $i = 1, 2, ..., n$, such that

$$f_i(0, 0) = 0 \tag{52}$$

$$\frac{\partial V_i}{\partial x_i} f_i(x, r_i) \leq -\alpha_i(V_i(x_i)) + \max\left\{ \max_{j\in\{1,2,...,n\}} \sigma_{ij}(V_j(x_j)),\ \kappa_i(|r_i|) \right\}, \quad \forall x \in \mathbb{R}^N,\ r_i \in \mathbb{R}^{K_i} \tag{53}$$

$$\left.\begin{array}{l} V_i(x_i) = V_i(\overline{x}_i),\ |r_i| \leq |\overline{r}_i| \\ V_j(x_j) \leq V_j(\overline{x}_j),\ \forall j \neq i \end{array}\right\} \Rightarrow \frac{\partial V_i}{\partial x_i} f_i(x, r_i) \leq \frac{\partial V_i}{\partial x_i} f_i(\overline{x}, \overline{r}_i) \tag{54}$$

$$\left.\begin{array}{l} (1+\delta)\alpha_i(V_i(x_i)) \leq \max\left\{ \max_{j\in\{1,2,...,n\}} \sigma_{ij}(V_j(x_j)),\ \kappa_i(|r_i|) \right\} \\ \epsilon_i \leq V_i(x_i)\ \vee\ x_i = 0 \\ \epsilon_i \leq |r_i|\ \vee\ r_i = 0 \end{array}\right\} \Rightarrow \frac{\partial V_i}{\partial x_i} f_i(x, r_i) \geq \delta_i \alpha(V_i(x_i)) \tag{55}$$

hold, where $x = [x_1^T, x_2^T, ..., x_n^T]^T \in \mathbb{R}^N$ and $x_i \in \mathbb{R}^{N_i}$.

This lemma is proved by partially modifying the argument used for the summation case [8], and incorporating the exogenous signal r_i into f_i. The above lemma establishes the property (54) which is absent in [8]. Subsystems Σ_i achieving (52)-(55) can be constructed as follows: Pick $p \geq 1$, $b > 0$ and $L > 1$ such that $\alpha_i \in \mathcal{O}(p)$ and $\sigma_{ij} \in \mathcal{O}(b)$ hold for $i, j = 1, 2, ..., n$, and $(1/Lp) + (1/Lb) < 1$. Let $Q \geq 1$

be such that $\kappa_i \in \mathcal{O}(Lb/Q)$. Define $\breve{\sigma}_{ij}(s) = \sigma_{ij}(s^L)$ and

$$\breve{\kappa}_i(s) = \begin{cases} \kappa_i(\epsilon_i)\left(\kappa_i(s)/\kappa_i(\epsilon_i)\right)^Q, & \text{for } s \in [0, \epsilon_i) \\ \kappa_i(s), & \text{for } s \in [\epsilon_i, \infty). \end{cases}$$

In the case of $p > 1$, let $\hat{\alpha}_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $q > 1$ be such that $\alpha_i(|x_i|^L) = \hat{\alpha}_i(|x_i|)|x_i|^{Lp}$ and $(1/p) + (1/q) = 1$. Define $V_i(x_i) = |x_i|^L$ and

$$\begin{aligned} f_i(x, r_i) = &\frac{-(q(1+\delta)+p)}{Lp}\hat{\alpha}_i(|x_i|)|x_i|^{Lp/q}x_i \\ &+ \frac{1}{L}\left\{q(1+\delta)\hat{\alpha}_i(|x_i|)\right\}^{1/p} \cdot \\ &\left\{q \max\left\{\max_{j\in\{1,2,\ldots,n\}} \breve{\sigma}_{ij}(|x_j|),\, \breve{\kappa}_i(|r_i|)\right\}\right\}^{1/q} x_i \end{aligned}$$

for $i = 1, 2, ..., n$. Then (52)-(55) are fulfilled for $\epsilon_i = 0$. In the $p = 1$ case, pick $\epsilon_i \in [0, \bar{\epsilon}_i]$. Choose $\tilde{p} \in (1, 2]$ and $q > 1$ such that $(1/L\tilde{p}) + (1/Lb) < 1$ and $(1/\tilde{p}) + (1/q) = 1$. Using

$$\alpha_{Ai}(|x_i|^L) = \hat{\alpha}_{Ai}(|x_i|)|x_i|^L,\ \alpha_{Ai}(s) = \begin{cases}\left\{1 - \frac{s}{\epsilon_i}\right\}\alpha_i(s), & s \in [0, \epsilon_i) \\ 0, & s \in [\epsilon_i, \infty)\end{cases}$$

$$\alpha_{Bi}(|x_i|^L) = \hat{\alpha}_{Bi}(|x_i|)|x_i|^{L\tilde{p}},\ \alpha_{Bi}(s) = \begin{cases} s\alpha_i(s)/\epsilon_i, & s \in [0, \epsilon_i) \\ \alpha_i(s), & s \in [\epsilon_i, \infty)\end{cases}$$

we can achieve (52)-(55) by $V_i(x_i) = |x_i|^L$ and

$$\begin{aligned} f_i = &\frac{-1}{L}\hat{\alpha}_{Ai}(|x_i|)x_i + \frac{-(q(1+\delta)+\tilde{p})}{L\tilde{p}}\hat{\alpha}_{Bi}(|x_i|)|x_i|^{L\tilde{p}/q}x_i \\ &+ \frac{1}{L}\left\{q(1+\delta)\hat{\alpha}_{Bi}(|x_i|)\right\}^{1/\tilde{p}} \cdot \\ &\left\{q \max\left\{\max_{j\in\{1,2,\ldots,n\}} \breve{\sigma}_{ij}(|x_j|),\, \breve{\kappa}_i(|r_i|)\right\}\right\}^{1/q} x_i. \end{aligned}$$

VIII. Concluding Remarks

In this paper, necessary and sufficient conditions have been developed for internal and external stability of networks whose subsystems are not necessarily ISS. Subsystems are formulated with supply rates taking the maximum of coupling signals (4). To the best of the authors' knowledge, the maximization supply rates have not been employed in the previously available stability criteria for iISS networks. The previous result [12] proposing an iISS small-gain criterion and a construction of Lyapunov functions of networks was based on the summation formulation of supply rates for subsystems (3). Taking the maximum formulation of supply rates, this paper has shown the two points regarding sufficient conditions: 1) The step of covering by subgraphs on which the previous result in [12] relies can be removed; 2) The small-gain criterion is equivalently expressed by matrix-like conditions generalizing an ISS result [6]. Note that, as in the ISS case [20], [17], [4], Item 2 has been achieved only for the maximum formulation of iISS supply rates. Necessary conditions have also been obtained for the stability of iISS network defined with the maximum supply rates: A) The necessity results are parallel to the previous results for summation supply rates [8], [9], [10]; B) The maximization involving external signals brings in a fundamental difference in the required small-gain margin.

All results in this paper can be repeated for the formulation which replaces $\{\alpha_i(V_i(x_i)), \sigma_{i,j}(V_j(x_j))\}$ and $\alpha_i \in \mathcal{O}(1)$ in (7) with $\{\alpha_i(|x_i|), \sigma_{i,j}(|x_j|)\}$ and $\alpha_i \in \mathcal{O}(>1)$.

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