Max- and Sum-Separable Lyapunov Functions for Monotone Systems and Their Level Sets

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***Abstract*— For interconnected systems and systems of large size, aggregating information of subsystems studied individually is useful for addressing the overall stability. In the Lyapunov-based analysis, summation and maximization of separately constructed functions are two typical approaches in such a philosophy. This paper focuses on monotone systems which are common in control applications and elucidates some fundamental limitations of max-separable Lyapunov functions in estimating domains of attractions. This paper presents several methods of constructing sum- and max-separable Lyapunov functions for second order monotone systems, and some comparative discussions are given through illustrative examples.**

I. Introduction

Monotone systems for which trajectories preserve a partial ordering on states forms an important class of dynamical systems in various fields of science and engineering such as biology, logistics and chemical processes [20], [11], [16]. It is also known that monotone systems play a key role in stability analysis of interconnected systems in the framework of integral input-to-state stability (iISS) [22]. In fact, the stability problem can be recast as a stability problem of a lower-dimensional, monotone comparison system [1], [15], [19], [4]. To study stability of monotone systems, typical Lyapunov functions employed in the recent literature (e.g.[4], [7], [8], [6], [9], [3], [19], [12], [13]) are in the form of

$$V(x) = \sum_i V_i(x_i) \tag{1}$$

or

$$V(x) = \max_i V_i(x_i), \tag{2}$$

where $x = [x_1, x_2, ..., x_n]$ is the state of the overall monotone system, while x_i is the state of i-th one-dimensional subsystem. In [14], the former is referred to as a sum-separable Lyapunov function, and the latter is referred to as a max-separable Lyapunov function. For an interconnection of iISS subsystems, it has been proved in [6] that the max-separable Lyapunov function (2) cannot guarantee global asymptotic stability of the interconnected system if the subsystems are only assumed to be iISS. The max-separable Lyapunov function can guarantee the stability only if all subsystems have a stronger property called input-to-state stability (ISS) [21], which is a special case of iISS [2]. On the other hand, it was recently reported in [14] that globally asymptotically stable monotone systems can always have a max-separable Lyapunov function on every compact invariant set. It is of course true that global asymptotic stability does not at all imply ISS of one-dimensional subsystems and it can allow subsystems to be merely iISS. Thus, as stated in [14], the compactness assumption must be essential for the construction of a max-separable Lyapunov function. One of the purposes of this paper is to give an insight into the capability of max-separable Lyapunov functions constructed on compact state spaces.

A positive definite function is said to be a Lyapunov function if its time derivative along the solutions of a system is non-positive in a neighborhood of an equilibrium one focuses on [10]. Negativity of the time derivative and the region where the negativity holds become interesting and important issues when one wants to establish asymptotic stability of the equilibrium and to estimate the domain of attraction. A Lyapunov function provides us with an estimate of the domain of attraction as a level set defined by the region where the Lyapunov function is less than a given level [10]. By studying closely the level sets, this paper resolves the seemingly-contradictory results on max-separable Lyapunov functions reported in [14], [6]. In this work, we elucidate the mechanism of max-separable Lyapunov functions whose level sets cannot be larger than a certain size even if the origin is globally asymptotically stable. A max-separable Lyapunov function helps one understand system behavior in a domain given arbitrarily when it is combined with boundedness or invariant sets deduced separately from monotonicity, conservation laws or physical restrictions. In addition to max-separable Lyapunov functions, this paper presents several methods of constructing sum-separable Lyapunov functions for second order monotone systems and demonstrates the additional requirements enabling us to construct such Lyapunov functions compared with max-separable ones. Due to space limitations, all proofs are omitted.

II. System and Notation

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. We also use the extended space $\overline{\mathbb{R}}_+ = [0, \infty]$. For n-dimensional vectors $x, y \in \mathbb{R}^n_+$, the component-wise partial order $x \leq y$ is meant by $x_i \leq y_i$ for all i. We write $x < y$ if $x \leq y$ but $x \neq y$. We write $x \ll y$ if $x_i < y_i$ holds for all i. A function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{P}$ and written as $\omega \in \mathcal{P}$ if ω is continuous and satisfies $\omega(0) = 0$ and $\omega(s) > 0$ for all $s \in \mathbb{R}_+ \setminus \{0\}$. A function $\omega \in \mathcal{P}$ is

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said to be of class $\mathcal{K}$ if ω is strictly increasing. A class $\mathcal{K}$ function is said to be of class $\mathcal{K}_\infty$ if it is unbounded. For $\omega \in \mathcal{K}$, the function $\omega^\ominus$: $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is defined as $\omega^\ominus(s) = \sup\{v \in \mathbb{R}_+ : s \geq \omega(v)\}$. By definition, $\omega^\ominus(s) = \infty$ for $s \geq \lim_{\tau\to\infty} \omega(\tau)$, and $\omega^\ominus(s) = \omega^{-1}(s)$ elsewhere. A map $\omega : \mathbb{R}^n_+ \to \mathbb{R}_+$ is extended to a map $\overline{\mathbb{R}}^n_+ \to \overline{\mathbb{R}}_+$ by letting $\omega(x) := \sup_{\{y\in\mathbb{R}^n_+ : y\leq x\}} \omega(y)$.

This paper considers systems of the form

$$\dot{x} = f(x) \tag{3}$$

where the function $f = [f_1^T, f_2^T]^T : \mathbb{R}^2_+ \to \mathbb{R}^2$ is locally Lipschitz and $f(0) = 0$. The unique maximal solution guaranteed by this local Lipschitzness is denoted by $\varphi(t, x_0)$, where $x_0 \in \mathbb{R}^2_+$ is the initial condition given at $t = 0$. We assume that system (3) is monotone, i.e., $x \leq y$ implies $\varphi(t, x) \leq \varphi(t, y)$ for all $t > 0$ in the maximal interval of existence. This property is guaranteed if and only if

$$x \leq y \text{ and } x_i = y_i \ \Rightarrow\ f_i(x) \leq f_i(y). \tag{4}$$

holds [11], [20][1]. Solutions of (3) starting in $\mathbb{R}^2_+$ must remain therein as long as they exist, due to (4). The origin $x = 0$ of (3) is said to be globally asymptotically stable (GAS) if it is stable in the sense of Lyapunov, and $\lim_{t\to\infty} \varphi(t, x_0) = 0$ holds for all $x_0 \in \mathbb{R}^2$. Note that attractivity implies the stability of the origin for monotone systems (see e.g. [19]). In this paper, a set $X \subseteq \mathbb{R}^2_+$ is said to be a domain of attraction if $\lim_{t\to\infty} \varphi(t, x_0) = 0$ holds for all $x_0 \in X$. Note that, in the literature, such a set X is sometimes referred to as an estimate of the domain of attraction. This paper drops "estimate of the" for brevity on the premise that the purpose of this paper is to obtain such a region as large as possible.

III. Limitations of Max-Separable Lyapunov Candidates

We shall begin with some main results in this section.

A. Global Asymptotic Stability

Max-separable Lyapunov functions possess the following fundamental property when their derivative is rendered negative in the entire state space.

Theorem 1: Suppose that there exists $\eta \in \mathcal{K}$ such that the implication

$$\eta(x_1) \leq x_2 \ \Rightarrow\ f_1(x) \geq 0 \tag{5}$$

holds for all $x \in \mathbb{R}^2_+$. If there exist differentiable functions $\rho_1, \rho_2 \in \mathcal{K}$ such that

$$V(x) = \max\{\rho_1(x_1), \rho_2(x_2)\} \tag{6}$$

satisfies

$$\frac{\partial V}{\partial x} f(x) < 0, \quad \forall x \in \mathbb{R}^2_+ \setminus \{0\} \tag{7}$$

(at differentiable points), then it holds that

$$\lim_{s\to\infty} \rho_1(s) \leq \lim_{s\to\infty} \rho_2 \circ \eta(s). \tag{8}$$

[1]Functions f satisfying (4) is often said to be quasi-monotone nondecreasing or type K.

Inequality (8) imposes a serious constraint on level sets of the Lyapunov candidate V in (6). Recall that a Lyapunov candidate V whose time derivative is negative along all solutions to (3) provides information about stability with the help of its level sets. In other words, a Lyapunov candidate V yields domains of attraction in terms of level sets, and the level sets are "basins" of attraction. To clarify the influence of (8) on domains of attraction predictable via (6), for each $l \in \mathbb{R}_+$, define the level set

$$L(l) := \left\{x \in \overline{\mathbb{R}}^2_+ : V(x) \leq l\right\} \tag{9}$$

associated with a given positive definite continuous function $V : \mathbb{R}^2_+ \to \mathbb{R}_+$. It is stressed that this paper allows $L(l)$ to be called a level set even if it contains ∞, i.e. $L(l) \not\subseteq \mathbb{R}^2_+$. Level sets $L(l)$ are bounded for all $l \in \mathbb{R}_+$ if and only if V is radially unbounded. A level set $L(l)$ is bounded if and only if $L(l) \subseteq \mathbb{R}^2_+$. In terms of the size of level sets, the next proposition derived from (8) highlights the limited capability of V in the form of (6).

Proposition 1: Let η be a class $\mathcal{K}$ function satisfying (5) for all $x \in \mathbb{R}^2_+$. Suppose that there exist differentiable functions $\rho_1, \rho_2 \in \mathcal{K}$ satisfying (7) with (6). If

$$\lim_{s\to\infty} \eta(s) < \infty \tag{10}$$

holds, every level set $L(l)$ containing a point $x \in \mathbb{R}^2_+$ satisfying $x_2 \geq \lim_{s\to\infty} \eta(s)$ is unbounded.

The unboundedness of the level sets proved in the above proposition implies that the max-separable Lyapunov function (6) cannot ensure the boundedness of solutions to (3) for all $x(0) \in \mathbb{R}^2_+$ if there exists $\eta \in \mathcal{K} \setminus \mathcal{K}_\infty$ satisfying (5). The unboundedness arises from radial boundedness of V resulting from (8) in the case of $\eta \in \mathcal{K} \setminus \mathcal{K}_\infty$. One may argue that radially unboundedness of V is not important if we are interested only in compact domains instead of the global one (7). The next section addresses this issue.

B. Asymptotic Stability on Compact Sets

What can max-separable Lyapunov functions say about stability when we are only interested in compact sets in $\mathbb{R}^2_+$? The following demonstrates explicitly a limitation of max-separable Lyapunov functions defined on compact sets.

Theorem 2: Let η be a class $\mathcal{K}$ function satisfying (5) for all $x \in \mathbb{R}^2_+$. If there exist differentiable functions $\rho_1, \rho_2 \in \mathcal{K}$ and a real number $l > 0$ such that

$$L(l) \subseteq \mathbb{R}^2_+ \tag{11}$$

$$\frac{\partial V}{\partial x}(x) f(x) < 0, \quad \forall x \in L(l) \setminus \{0\} \tag{12}$$

at differentiable points of V in (6), then it holds that

$$\rho_2^{-1}(l) < \eta \circ \rho_1^{-1}(l) < \infty. \tag{13}$$

A level set $L(l)$ achieving (12) is a domain of attraction if it is compact, i.e., (11). Recall that $L(l)$ is bounded if and only if (11). The necessary condition (13) in Theorem 2 indicates that no max-separable function V estimates a domain of attraction allowing its x_2-component to be larger than or equal to $\lim_{s\to\infty}\eta(s)$. In other words, $L(l) \subseteq \mathbb{R}_+ \times [0, \lim_{s\to\infty}\eta(s))$ is necessary for V to ensure the boundedness of $x_1(t)$. Therefore, when η in (5) is bounded, domains of attraction provable by max-separable Lyapunov functions are limited by the finite value $\lim_{s\to\infty}\eta(s)$ in the x_2-component. This limitation is independent of how a max-separable Lyapunov function is constructed.

Remark 1: For monotone systems, there is a way to estimate domains of attraction [17], [18], [14] even if level sets of a Lyapunov function do not provide us with sufficiently large domains of attraction. If there exists $x_0 \in \mathbb{R}^2_+$ such that $\lim_{t\to\infty}\varphi(t, x_0) = 0$, then from the monotonicity of (3) it follows that the set

$$B(x_0) := \{x \in \mathbb{R}^2_+ : \exists t \in \mathbb{R}_+ \text{ s.t. } x \le \varphi(t, x_0)\}$$

is a domain of attraction and positively invariant. Here, $B(x_0)$ may not be a level set of any Lyapunov function at all.

IV. Examples of Max-Separable Lyapunov Functions

This section presents some examples of max-separable Lyapunov functions. In addition to their useful features, the mechanism of limitations of each construction in estimating domains of attraction is illustrated.

Method 1 Rantzer et al. [14] proposed a way to construct a Lyapunov function on compact sets for (3). The construction assumes GAS of $x = 0$ and the existence of a positively invariant set X. For such a set X given arbitrarily, define a vector $\overline{x} \in \mathbb{R}^2_+$ outside X as

$$\overline{x}_i = 1 + \sup\{x_i \in \mathbb{R}_+ : x \in X\}, \quad i = 1, 2 \qquad (14)$$
$$\overline{x} = [\overline{x}_1, \overline{x}_2]^T.$$

The most useful and unique point of the construction V in [14] is that it uses only the temporal information of the single trajectory $\varphi(t, \overline{x})$ of (3) as

$$\rho_i(x_i) = e^{-T_i(x_i)}, \quad i = 1, 2 \qquad (15)$$
$$T_i(x_i) := \max\{\tau \in \mathbb{R}_+ : x_i \le \varphi_i(t, \overline{x}), \forall t \in [0, \tau]\}. \quad (16)$$

By virtue of the monotonicity of (3), the function V in (6) with (15) achieves $\partial V/\partial x \cdot f < 0$ for all differentiable points $x \in X \setminus \{0\}$. It gives us domains of attraction as $L(l) \subseteq X$ for $l \in [0, l_{max}]$, where $l_{max} = \max\{l \in \mathbb{R}_+ : L(l) \subseteq X\}$. To see that this construction is consistent with Theorems 1, 2 and Proposition 1, suppose that exists $\eta \in \mathcal{K}$ satisfying (5) for all $x \in \mathbb{R}^2_+$. If $p = [p_1, p_2]^T \in X \setminus \{0\}$ satisfies $\eta(p_1) \le p_2$, then definition (16) and property (5) yield $T_1(p_1) > T_2(p_2)$, i.e., $\rho_1(p_1) < \rho_2(p_2)$ in (15). If $\eta \in \mathcal{K} \setminus \mathcal{K}_\infty$ and $\lim_{s\to\infty}\eta(s) \le p_2$ hold, then $\rho_1(p_1) < \rho_2(p_2)$ holds for any $p_1 \in \mathbb{R}_+$. Thus, we have (8) and (13).

Remark 2: One may not want to stick to level sets, as long as the trajectory $\varphi_i(t, \overline{x})$ is computed. For example, the set $J(l) := \{x \in B(\overline{x}) : V(x) \le l\}$ is a domain of attraction and positively invariant. However, if there exists $\eta \in \mathcal{K}\setminus\mathcal{K}_\infty$ satisfying (5) for all $x \in \mathbb{R}^2_+$, $T_i(x_i)$ is not defined by (16) for $x_i > \bar{x}_i$. Thus, V is not guaranteed to be defined on $J(l)$. Note that the GAS assumption implies that without referring to V, any bounded set is a domain of attraction, although it may not be positively invariant.

Method 2 Exploiting the idea proposed for ISS systems in [9], we can also use a single trajectory to construct another Lyapunov function providing a domain of attraction as large as the construction by Rantzer et al. [14]. This time, we use the configurational information of the trajectory. For system (3), assume again that $x = 0$ is GAS. Define

$$\Omega = \{x = [x_1, x_2]^T \in \mathbb{R}^2_+ : f(x) \ll 0\}. \qquad (17)$$

The origin $x = 0$ of (3) is GAS only if the set Ω divides the positive open orthant $(0, \infty)^2$ into two disjoint sets or the closure of Ω contains either $\mathbb{R}_+ \times \{0\}$ or $\{0\} \times \mathbb{R}_+$ (See e.g. [19], [17]). Furthermore, the set Ω is positively invariant [19], [20]. Let $\overline{x} = [\overline{x}_1, \overline{x}_2]^T \in \Omega$, which implies $\overline{x}_2 > 0$ (An algorithm for computing such a point $\overline{x} \in \mathbb{R}^n_+$ for general $n \ge 2$ is presented in [17].). Obviously, $\varphi(t, \overline{x}) \in \Omega$ holds for all $t \in \mathbb{R}_+$. Then the set

$$\{x \in \mathbb{R}^2_+ : \exists t \in \mathbb{R}_+ \text{ s.t. } x = \varphi(t, \overline{x})\} \cup \{0\}$$

can be considered as the graph $(x_1, \omega(x_1))$ of a function $\omega : [0, \overline{x}_1] \to \mathbb{R}_+$. Due to the definition of Ω and the differentiability of $\varphi_i(t, \overline{x})$ in t for $i = 1, 2$, the set ω is guaranteed to be strictly increasing, to satisfy $\omega(0) = 0$ and to be differentiable except at the origin. Suppose that ω is differentiable at the origin. Define V as (6) with

$$\rho_1(x_1) = \omega(x_1), \qquad x_1 \in [0, \overline{x}_1] \qquad (18)$$
$$\rho_2(x_2) = x_2, \qquad x_2 \in \mathbb{R}_+. \qquad (19)$$

Let $D = [0, \overline{x}_1] \times \mathbb{R}_+$. The definitions of ρ_1, ρ_2 and Ω directly yield $\partial V/\partial x \cdot f < 0$ for all differentiable points $x \in D \setminus \{0\}$. A domain of attraction is obtained as $L(\overline{x}_2) = [0, \overline{x}_1] \times [0, \overline{x}_2]$. Suppose that there exists $\eta \in \mathcal{K}$ satisfying (5) for all $x \in \mathbb{R}^2_+$. From the definition of Ω, we have $\omega(s) \le \eta(s)$ for all $s \in [0, \overline{x}_1]$. Then the choice of ω directly yields $\overline{x}_2 < \eta(\overline{x}_1)$. From (19) and the definition of ω we obtain $\rho_2^{-1}(\overline{x}_2) < \eta \circ \omega^{-1}(\overline{x}_2) < \infty$. Using $l = \overline{x}_2$ and (18) we arrive at (13). If $\eta \in \mathcal{K} \setminus \mathcal{K}_\infty$ holds in addition, due to the definition of ω and (18)-(19), every level set $L(l)$ containing a point x satisfying $x_2 \ge \lim_{s\to\infty}\eta(s)$ becomes unbounded. Moreover, from $\eta \in \mathcal{K} \setminus \mathcal{K}_\infty$ and the definition of Ω it follows that $\rho_2^{-1}(l) < \lim_{s\to\infty}\eta(s)$ as long as $l \in [0, \lim_{s\to\infty}\rho_1(s))$. This results in (8). Therefore, the construction of V with (18) and (19) is consistent with Theorems 1, 2 and Proposition 1.

Method 3 The path $\omega(x_1)$ does not have to be generated by the trajectory $\varphi(t, \overline{x})$. Let $(x_1, \omega(x_1))$ be the graph of a function $\omega \in \mathcal{K}$ and contained in Ω. Then the function V defined in (6) with

$$\rho_1(x_1) = \omega(x_1), \qquad x_i \in \mathbb{R}_+ \qquad (20)$$
$$\rho_2(x_2) = x_2, \qquad x_2 \in \mathbb{R}_+. \qquad (21)$$

achieves $\partial V/\partial x \cdot f < 0$ for all differentiable points $x \in \mathbb{R}^2_+ \setminus \{0\}$. The level set $L(l)$ is bounded and a domain of attraction if $l < \lim_{s\to\infty}\omega(s)$. Therefore, the pair (20) and (21) complies with Theorems 1 and Proposition 1.

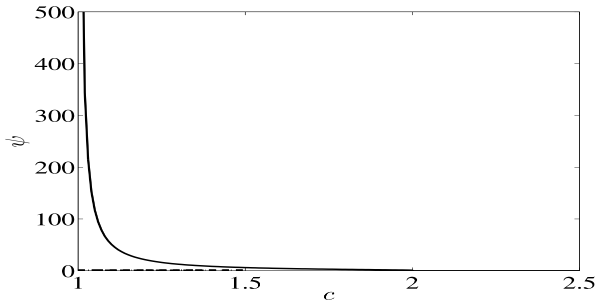

Fig. 1. Smallest exponent ψ in a sum-separable Lyapunov function.

Remark 3: The pair of $\rho_1(x_1) = x_1$ and $\rho_2(x_2) = \omega^{-1}(x_2)$ in (6) also gives a Lyapunov function defined on $\mathbb{R}_+ \times [0, \overline{x}_2]$. The function V becomes infinity for large x_2 for which max-separable Lyapunov functions cannot deal with if a function $\eta \in \mathcal{K} \setminus \mathcal{K}_\infty$ satisfies (5) for all $x \in \mathbb{R}^2_+$. However, the Lyapunov function can provide the domain of attraction as $L(\overline{x}_1) = [0, \overline{x}_1] \times [0, \overline{x}_2]$ that is the same as the one obtained with (18) and (19).

Remark 4: The utilization of a single trajectory for verifying stability and robustness of monotone systems is pursued in [17], [18].

Remark 5: GAS of $x = 0$ of (3) implies that at least one of x_1- and x_2-systems in (3) is ISS with respect to input x_{3-i}. This fact follows from the existence of the set Ω by virtue of the monotonicity of (3). If $\eta \in \mathcal{K}_\infty$ holds, x_1-system is ISS [22], [2].

Remark 6: All methods in this section yield max-separable Lyapunov functions in a compact set if the assumption of GAS is replaced by attractivity of $x = 0$ in a compact domain of interest. The limitation of max-separable Lyapunov functions exhibits unless the domain of interest is below the limitation characterized as $x_2 < \lim_{s\to\infty} \eta(s)$.

V. Examples of Sum-Separable Lyapunov Functions

It is not yet known that sum-separable Lyapunov functions can be generated directly from a single trajectory. However, it is possible to analytically construct a sum-separable Lyapunov functions based on information little more than the positively invariant set Ω.

Method 4 Consider the monotone system in the form of

$$\dot{x}_i = -\alpha_i(x_i) + \sigma_i(x_{3-i}), \quad \alpha_i \in \mathcal{K}, \quad \sigma_i \in \mathcal{K}. \qquad i = 1, 2 \tag{22}$$

The monotonicity of (22) is clear since (4) holds. The representation (22) provides more information than the Ω used in the previous subsection. At the price of this extra information, a sum-type construction allows us get rid of the limitations described by Theorems 1, 2 and Proposition 1.

Remark 7: For each $i = 1, 2$, x_i-system in (22) is iISS with respect to input x_{3-i} [2]. It becomes ISS if and only if $\lim_{s\to\infty} \sigma_i(s) \le \lim_{s\to\infty} \alpha_i(s)$ holds [23].

As demonstrated in [1], unless $\lim_{s\to\infty} \sigma_i(s) \le \lim_{s\to\infty} \alpha_i(s)$ holds for both $i = 1, 2$, the existence of Ω dividing $(0, \infty)^2$ into two disjoint sets is not sufficient for guaranteeing GAS of (22)[2]. For ensuring the asymptotic stability in the global sense, it is sufficient that the "width" of Ω does not shrink to zero toward the radial direction. In fact, we can verify the following by making use of the argument in [8], where $c > 1$ prevents Ω from shrinking to zero.

Proposition 2: Suppose that there exists $c > 1$ such that

$$\alpha_1^{\ominus} \circ c\sigma_1 \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \le s, \quad \forall s \in \mathbb{R}_+ \tag{23}$$

holds[3]. Then the continuously differentiable function $V : \mathbb{R}^2_+ \to \mathbb{R}_+$ defined by

$$V(x) = \rho_1(x_1) + \rho_2(x_2) \tag{24}$$

$$\rho_i(s) = \int_0^s \lambda_i(\tau) d\tau, \quad i = 1, 2 \tag{25}$$

$$\lambda_i(s) = \alpha_i(s)^{\psi} \sigma_{3-i}(s)^{\psi+1}, \ i = 1, 2 \tag{26}$$

is positive definite, radially unbounded and achieves (7), where $\psi \ge 0$ is a real number satisfying

$$\begin{cases} \psi = 0 & \text{if } c > 2 \\ \psi^{-\frac{\psi}{\psi+1}} < \dfrac{c}{\psi+1} \le 1 & \text{otherwise.} \end{cases} \tag{27}$$

Since $\lambda_i \in \mathcal{K}$ holds for $i = 1, 2$, we have $\rho_i \in \mathcal{K}_\infty$, $i = 1, 2$. Thus, the level set $L(l)$ defined as in (9) is a positively invariant set being a domain of attraction for any $l > 0$, and the sum-separable Lyapunov function V in (24) establishes GAS of the origin $x = 0$ for (22). Note that there always exists $\psi \ge 0$ satisfying (27). In fact, $c \le \psi + 1$ is met for a sufficiently large $\psi \ge 0$, and we have

$$\lim_{\psi\to\infty} (\psi + 1)\psi^{-\frac{\psi}{\psi+1}} = 1.$$

Due to $c > 1$ and continuity, the requirement (27) is achieved by a sufficiently large $\psi \ge 0$. The smallest ψ achieving (27) is computed for each $c > 1$ and plotted in Fig.1.

Method 5 The choice of the pair ρ_i, $i = 1, 2$ establishing GAS of $x = 0$ of (22) is not unique. Suppose that there exist $c_i > 0$, $i = 1, 2$, and $k > 0$ such that

$$\sigma_2(s)^k \le c_1 \alpha_1(s), \quad \forall s \in \mathbb{R}_+ \tag{28}$$

$$c_2 \sigma_1(s) \le \alpha_2(s)^k, \quad \forall s \in \mathbb{R}_+ \tag{29}$$

$$c_1 < c_2 \tag{30}$$

hold. This triplet is a sufficient condition for the existence of $c > 1$ satisfying (23), but it is not necessary. The existence of $c_1, c_2, k > 0$ satisfying (28)-(30) allows us to replace (26) by another formula for constructing a Lyapunov function via (24)-(25). In fact, according to [4], in the case of $k \ge 1$, we can verify that the pair

$$\lambda_1 = c_1 \left(\frac{c_2}{c_1}\right)^{\frac{k+1}{k+2}}, \quad \lambda_2(s) = k\alpha_2(s)^{k-1} \tag{31}$$

achieves (7). For $k < 1$, the above pair is replaced by

$$\lambda_1(s) = \frac{1}{k}\alpha_1(s)^{\frac{1-k}{k}}, \quad \lambda_2 = c_1^{-\frac{1}{k}} \left(\frac{c_1}{c_2}\right)^{\frac{1}{1+2k}} \tag{32}$$

[2]The existence of Ω dividing $(0, \infty)^2$ into two disjoint sets is sufficient if we are only interested in compact domains of attraction.

[3]Condition (23) is called a small-gain condition [1], [4], [7].

The functions V in (24) constructed with both (31) and (32) are positive definite and radially unbounded.

Remark 8: For systems having $\sigma_i = 0$ in (22), replace $\sigma_i = 0$ with a new sufficiently small function $\sigma_i \in \mathcal{K}$ satisfying (23). Then applying **Method 4** and **Method 5** to the fictitious system with the new $\sigma_i \in \mathcal{K}$ gives us a Lyapunov function and domains of attraction for the original system.

Method 6 If system (22) admits a non-empty Ω, neither $\mathbb{R}_+ \times \{0\}$ nor $\{0\} \times \mathbb{R}_+$ is contained by the closure $\overline{\Omega}$ of Ω. To allow $\overline{\Omega}$ to contain $\mathbb{R}_+ \times \{0\}$, we consider

$$\begin{aligned} \dot{x}_1 &= -\alpha_1(x_1) + \sigma_1(x_2), \\ \dot{x}_2 &= -\alpha_2(x_2), \\ &\qquad \alpha_1, \alpha_2 \in \mathcal{P}, \quad \sigma_1 \in \mathcal{K}. \end{aligned} \tag{33}$$

Since the origin $x = 0$ of this system is not always GAS, we assume the existence of $k \geq 1$ such that

$$\int_1^\infty \alpha_1(s)^{k-1} ds = \infty, \quad \int_0^1 \frac{\sigma_1(s)^k}{\alpha_2(s)} ds < \infty \tag{34}$$

hold [5]. Following the argument in [5] we can verify that V in (24) constructed with

$$\lambda_1(s) = \frac{1}{2}\alpha_1(s)^{k-1} \tag{35}$$

$$\lambda_2(s) = \begin{cases} \dfrac{\sigma_1(s)^k}{\alpha_2(s)} & , s \in [0, 1) \\ \displaystyle\max_{w \in [1,s]} \frac{\sigma_1(w)^k}{\alpha_2(w)} & , s \in [1, \infty) \end{cases} \tag{36}$$

is positive definite, radially unbounded and achieves (7) for (33).

Remark 9: A class of systems that admit neither a sum-separable nor a max-separable Lyapunov function is suggested in [14, Eq.(9) with Fig.2]. The representations (22) and (33) exclude systems in such a class.

VI. Numerical Examples

In this section, we consider the system

$$\dot{x}_1 = -x_1 + x_1 x_2 \tag{37}$$

$$\dot{x}_2 = -2x_2 - x_2^2 + 2g(x_1) + g(x_1)^2, \tag{38}$$

where g is a class $\mathcal{K}$ function satisfying

$$\overline{g} := \lim_{s \to \infty} g(s) < 1. \tag{39}$$

This system (37)-(38) is monotone, and the set Ω in (17) is

$$\Omega = \{x \in \mathbb{R}_+^2 : g(x_1) < x_2 < 1\}.$$

Clearly, this set Ω divides $(0, \infty)^2$ into two disjoint sets. This fact together with (39) guarantees that the origin $x = 0$ of the system (37)-(38) is GAS [4], [1], [7]. We shall construct Lyapunov functions for this system via the several methods described in Sections IV and V and discuss their features.

Method 1 First, a positively invariant set X is computed numerically as X in Fig.2. For this set, we obtain $\overline{x} = [5.072, 2.3]^T$ from (14). The trajectory $\varphi(t, \overline{x})$ of (37)-(38) for $t \in \mathbb{R}_+$ gives the Lyapunov function V as in (6) with (15) and (16). Level sets $L(l)$ defined as (9) for the constructed function V are depicted for several values of $l > 0$ in Fig.2.

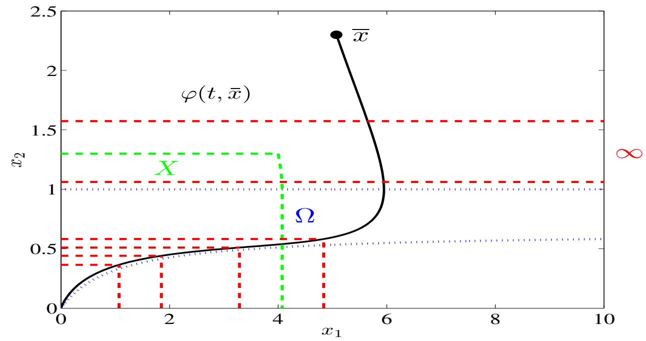

Fig. 2. Level sets of a max-separable Lyapunov function for (37)-(38) and (40) via **Method 1** with $l = 0.02, 0.05, 0.15, 0.35, 0.75, 0.9$.

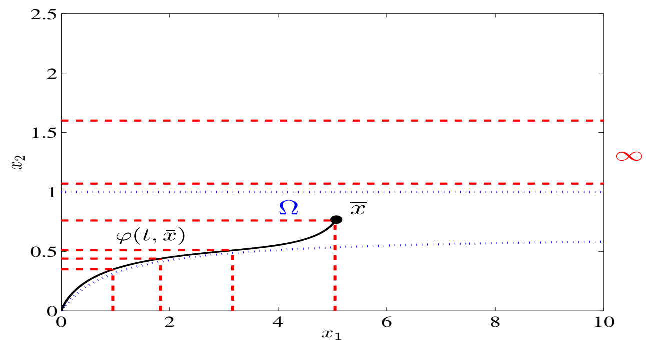

Fig. 3. Level sets of a max-separable Lyapunov function for (37)-(38) and (40) via **Method 2** with $l = 0.35, 0.44, 0.51, 0.76, 1.07, 1.6$.

The plots are computed for

$$g(x_1) = \frac{16x_1}{25(1 + x_1)}. \tag{40}$$

The level sets illustrated as dashed rectangles in red are domains of attraction deduced from the function V. The rectangles transform into horizontal lines of infinite length when their height exceeds unity. Hence, there is no $l > 0$ such that the level set $L(l)$ is bounded and contains $x_2 \geq 1$, although the invariant set X from which we started allows x_2 to be larger than 1. Thus, the set X is not contained in any bounded level set. In Fig.2, the red dashed lines are drawn unboundedly even in the region of $x_1 > \overline{x}_1$ for which ρ_1 is not defined. Indeed, since the Lyapunov function V provides no information about x_1-component there, it by itself does not guarantee the boundedness of x_1 for $x_2 \geq 1$.

Method 2 Pick $\overline{x}_1 = 5.072$ and define $\overline{x} = [\overline{x}_1, \overline{x}_2]^T \in \Omega$ with $\overline{x}_2 = (1 + g(\overline{x}_1))/2$. Define the function $\omega : [0, \overline{x}_1] \to \mathbb{R}_+$ so that the trajectory $\varphi(t, \overline{x})$ of (37)-(38) for $t \in \mathbb{R}_+$traces the graph $(x_1, \omega(x_1))$. Define V as in (6) with

$$\rho_1(s) = \omega(s), \qquad s \in [0, \overline{x}_1] \tag{41}$$

$$\rho_2(s) = s, \qquad s \in \mathbb{R}_+ \tag{42}$$

Domains of attraction computed as level sets $L(l)$ of the constructed V for (40) and several values of $l > 0$ are illustrated in Fig. 3. Although V is defined on $[0, \overline{x}_1] \times \mathbb{R}_+$, the rectangle $[0, \overline{x}_1] \times [0, \overline{x}_2]$ is the largest domain of attraction the Lyapunov function can yield. Note that $\overline{x}_2 < 1$ can be chosen arbitrarily close to unity for any given $\overline{x}_1 > 1$.

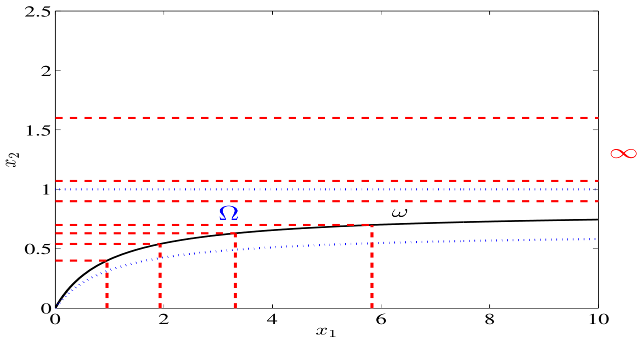

Fig. 4. Level sets of a max-separable Lyapunov function for (37)-(38) and (40) via **Method 3** with $l = 0.4, 0.54, 0.63, 0.7, 0.90, 1.07, 1.6$.

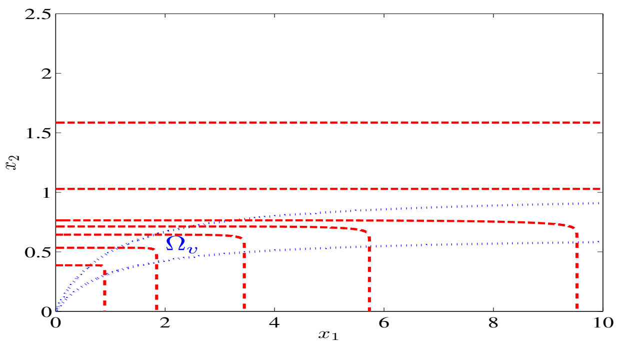

Fig. 5. Level sets of a sum-separable Lyapunov function for (37)-(38) and (40) via **Method 4** with $l = 10^{-10}, 3{\times}10^{-5}, 5{\times}10^{-2}, 3, 5{\times}10, 10^7, 10^{15}$.

<u>**Method 3**</u> Define

$$\rho_1(s) = \frac{1+\overline{g}}{2\overline{g}} g(s), \qquad s \in \mathbb{R}_+ \tag{43}$$

$$\rho_2(s) = s. \qquad s \in \mathbb{R}_+. \tag{44}$$

In contrast to (41)-(42), the above pair (43)-(44) defines V on the entire $\mathbb{R}^2_+$. However, as illustrated by the level sets $L(l)$ plotted for (40) in Fig.4, the function V in (6) cannot establish GAS for the origin of (37)-(38) since the x_2-component of bounded level sets cannot go beyond unity.

<u>**Method 4**</u> Consider the monotone system

$$\dot{v}_1 = -\hat{b}(v_1) + v_2 \tag{45}$$

$$\dot{v}_2 = -2v_2 - v_2^2 + 2\hat{g}(v_1) + \hat{g}(v_1)^2 \tag{46}$$

defined for $v = [v_1, v_2]^T \in \mathbb{R}^2_+$, where, for $s \in \mathbb{R}_+$,

$$b(s) = \frac{as}{1+as}, \ \hat{b}(s) = b\left(\frac{e^s-1}{a}\right), \ \hat{g}(s) = g\left(\frac{e^s-1}{a}\right).$$

Applying the diffeomorphisms $v_1 = \log(1+ax_1)$ and $v_2 = x_2$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $a > 0$ to (37) and (38) gives

$$\dot{v}_1 = -\frac{ax_1}{1+ax_1} + \frac{ax_1}{1+ax_1}x_2 \le -\hat{b}(v_1) + v_2$$

and (46). Thus, due to the standard argument of the comparison principle (e.g. [10], [11]), a domain of attraction of (45)-(46) is a domain of attraction of the original system (37)-(38). To use **Method 4** for constructing a Lyapunov function $V(v)$, take g in (40) and $a{=}1$. Then the functions

$$\alpha_1(s) = \hat{b}(s), \ \sigma_1(s) = s$$

$$\alpha_2(s) = 2s + s^2, \ \sigma_2(s) = 2\hat{g}(s) + \hat{g}(s)^2$$

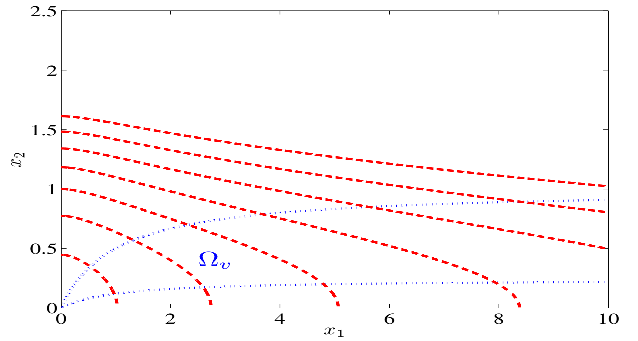

Fig. 6. Level sets of a sum-separable Lyapunov function for (37)-(38) and (50) via **Method 4** with $l = 0.1, 0.3, 0.5, 0.7, 0.9, 1.1, 1.3$.

satisfy (23) with $c \in (1, 5/4]$. Equations (26) and (27) with $c = 5/4$ lead to

$$\lambda_1(s) = \hat{b}(s)^{17}(2\hat{g}(s) + \hat{g}(s)^2)^{18}, \qquad s \in \mathbb{R}_+ \tag{47}$$

$$\lambda_2(s) = (2s+s^2)^{17}s^{18}, \qquad s \in \mathbb{R}_+. \tag{48}$$

From (24), a Lyapunov function V is obtained as

$$V(x) = \rho_1(v_1) + \rho_2(v_2) = \rho_1(\log(1+x_1)) + \rho_2(x_2) \tag{49}$$

with (25). Level sets $L(l)$ of (49) are plotted for several $l > 0$ in Fig.5, where

$$\Omega_v = \{x \in \mathbb{R}^2_+ : \alpha_i(v_i) > \sigma_i(v_{3-i}), \ i = 1, 2\}.$$

All level sets are compact and thus domains of attractions, although some parts of large x_1 exceed the frame of Fig.5. In fact, both ρ_1 and ρ_2 generated from (47) and (48) via (25) are radially unbounded, and so is V. This implies that for an arbitrarily large $x \in \mathbb{R}^2_+$, there always exists $l > 0$ such that $x \in L(l) \subseteq \mathbb{R}^2_+$. Therefore, the Lyapunov function (49) establishes GAS of $x = 0$. The level sets become more rounded and well-balanced in both x_1 and x_2 directions if Ω (or Ω_v) is wider. To see this, we replace (40) by

$$g(x_1) = \frac{6x_1}{25(1+x_1)}. \tag{50}$$

Property (23) is satisfied with $c = \sqrt{25/6}$. Then From $\psi = 0$ in (26) satisfying (27) it follows that

$$\lambda_1(s) = 2\hat{g}(s) + \hat{g}(s)^2, \qquad s \in \mathbb{R}_+ \tag{51}$$

$$\lambda_2(s) = s, \qquad s \in \mathbb{R}_+. \tag{52}$$

For (50) level sets $L(l)$ are plotted in Fig. 6. It may illustrate better than Fig. 5 that there always exists $l > 0$ such that $L(l)$ is large enough to contain $x \in \mathbb{R}^2_+$ given arbitrarily.

<u>**Discussion**</u> Although sum-separable Lyapunov are better than max-separable ones in being able to yield domains of attraction of unlimited size theoretically, the max-separable Lyapunov functions still have some advantages. In the above example, the max-separable constructions do not require the preprocess of computing α_i and σ_i from the original system equation. The functions ρ_i for the max-separable Lyapunov function (6) are independent of α_i and σ_i as long as the sign of $f_i(x)$ remains unchanged. For instance, the max-separable

Lyapunov functions V obtained above for the system (37)-(38) also achieve (12) for monotone systems such as

$$\dot{x}_1 = -x_1^5 + x_1^5 x_2 \tag{53}$$

$$\dot{x}_2 = (1 + x_2^2)(-x_2 + g(x_1)). \tag{54}$$

On the other hand, the sum-separable Lyapunov function (49) does not achieve (7) for the system (53)-(54). Another benefit of using max-separable Lyapunov functions is their handiness. The exponent ψ appearing in the sum-separable Lyapunov function is as large as in (47)-(48) and it is inherited from Fig.1 unless (23) holds with $c \geq 2$. In practice, the large exponent causes serious trouble in controller design based on Lyapunov functions when it results in very high "order" nonlinearities in controllers[4]. In addition, since (40) in (37)-(38) requires $c < 2$, when x_2 is allowed to be larger than unity, the level sets the sum-separable Lyapunov function produces became extremely large in x_1-direction, although they are guaranteed to remain bounded. The necessity of high order nonlinearities for sum-separable Lyapunov functions has not been proved. Nevertheless, except the special case of (28)-(30), we have not yet found ways to reduce the order when $c > 1$ in (23) needs to be close to unity. Compared with the max-separable Lyapunov functions, the order reduction is naturally harder since the transformation $W(V(x))$ by a class $\mathcal{K}$ function W destroys the sum-separability, while it preserves the max-separability.

VII. Concluding Remarks

This paper has presented several sum- and max-separable Lyapunov functions for monotone systems, and given comparative discussions from the aspect of domains of attraction predictable by the Lyapunov functions. It has been demonstrated that regardless of construction methods, max-separable Lyapunov functions have fundamental limitations in estimating domains of attraction. This paper has elucidated the circumstances where level sets of max-separable Lyapunov functions cannot be larger than a certain size even if the origin is GAS. In such a situation, it is possible to estimate a larger domain of attraction by making use of the monotonicity without relying solely on the level sets. Sum-separable Lyapunov functions do not suffer from such limitations. However, they require more information than max-separable Lyapunov functions, i.e., we need to separate variables in system equations to apply the formulas for constructing a Lyapunov function.

Unless we have an ideal matching between nonlinearities in the system equation, the exponent in the sum-separable Lyapunov function proposed by this paper becomes very large as the system approaches the stability margin very closely. Such a large exponent can deform the estimated invariant set unreasonably. The max-separable Lyapunov function switches between f_1 and f_2 by regions in terms of V. The sum-separable Lyapunov function switches between them in terms of the time-derivative of V. If the set Ω where the sign of f_1 and f_2 flips is narrow, the sharp interchange of the sign can be dealt with by ρ_i's whose derivative λ_i changes rapidly in magnitude to switch $|\lambda_1 f_1| > |\lambda_2 f_2|$ into $|\lambda_1 f_1| < |\lambda_2 f_2|$ and vice versa. This mechanism has resulted in the rectangular like estimates of domains of attraction in Fig.5 even for a sum-separable Lyapunov function. If the system has a wider stability margin, the estimated domains become rounded as in Fig.6.

[4]Whenever the convergence rate one wants to achieve by designing a controller is practically acceptable, the sum-separable construction does not generate any high-order nonlinearities. Property (23) requires c to be close to unity only when a controlled system has an "inaccessible" slow mode.

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