# Input-dependent stability analysis of systems with saturation in feedback

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*Abstract*— **The paper deals with global stability analysis of linear control systems with saturation in feedback driven by an external input. Various new criteria based on non-quadratic Lyapunov functions are proposed, that unlike many previous results, offer better account for the role of the external excitation by providing input-dependent conditions for stability of solutions. For example, it is shown that even if the system fails to satisfy the incremental version of the circle criterion, the stability is guaranteed whenever the uniform root mean square value of the input signal is less than a computable threshold. The general theoretical results are illustrated in the case of the double integrator closed by a saturated linear feedback with an external excitation.**

## I. Introduction

In most practical applications, control systems are affected by external inputs, either excitations or reference signals. The controllers are often to be designed so that all solutions of the resultant closed-loop system "forget" their initial states: they converge to a common process determined only by the input signal. For linear systems and feedbacks, this can be achieved by merely making the closed loop system asymptotically stable in the absence of inputs.

However, for nonlinear systems the situation is much more complicated. In particular, even if such system is globally asymptotically stable without inputs, a proper external input may exhibit unstable (yet bounded) behavior and/or multiple stable solutions. Conditions under which this is not the case and, moreover, all solutions converge to a common "steady-state" process have been addressed in a number of publications. The last property was investigated for systems with time-periodic right-hand sides in [1], where those with a unique periodic globally asymptotically stable solution were called *convergent*. This definition was extended to not necessarily time-periodic systems in [2]: a system is said to be *convergent* if there exists a unique solution that firstly is defined and bounded on the entire time axis and secondly is globally asymptotically stable; see also [3]. Since this solution attracts all other solutions, regardless of their initial states, it may be interpreted as a "steady-state" process. Convergence of solutions to each other was also addressed in the 60's by LaSalle and Lefschetz [4] and Yoshizawa [5].

Several decades after these publications, the interest in mutual attraction of solutions revived in control community. In the mid-nineties, [6] introduced the relevant notion of contraction and independently obtained the afore-mentioned result of Demidovich. A different approach was pursued in [7]. In [8], a Lyapunov approach was developed to study the relevant phenomenon of global uniform asymptotic stability of every solution, which was called *incremental stability*. Incremental stability is also known as extreme stability [5]. The differences between incremental stability and convergent systems are explored in [9].

Starting from [10], the overwhelming majority of constructive criteria for mutual convergence was established by means of quadratic Lyapunov functions. Furthermore, they give input-independent criteria that do not take into account particular properties of the inputs and thus often yield overly strong conditions: no input is capable of destroying the mutual convergence. To the best of the authors knowledge, few exceptions in the form of input-dependent criteria concern only very restricted classes of either systems [11] or inputs [12].

The goal of this paper is to derive new constructive input-dependent criteria for mutual convergence of solutions in systems with saturated feedback that provide a better insight on the relevant effect of the input. To this end, we employ non-quadratic Lyapunov functions by developing techniques recently presented in [13].

The paper is organized as follows. First we present a motivating example. Then, in Sect. III, we introduce the basic notion of convergent systems. The main ideas underlying our results are exposed in Sect. IV, whereas the main results are presented in Sect. V. They are illustrated by an example in Sect. VI. Finally, some conclusions are outlined. The proofs of the results are available upon request and will be given in the full version of this paper.

## II. Motivating example

We consider the double-integrator closed by the saturated linear feedback with an external harmonic excitation:

$$\ddot{x} = \mathrm{sat}\left[ -\dot{x} - x + u(t)\right],$$

where $u(t) = b\sin\omega t$. By employing a Lyapunov function in the Lur'e-Postnikov form (a quadratic form plus integral of the nonlinearity), it is straightforward to see that the free system ($u(t) \equiv 0$) is globally asymptotically stable.

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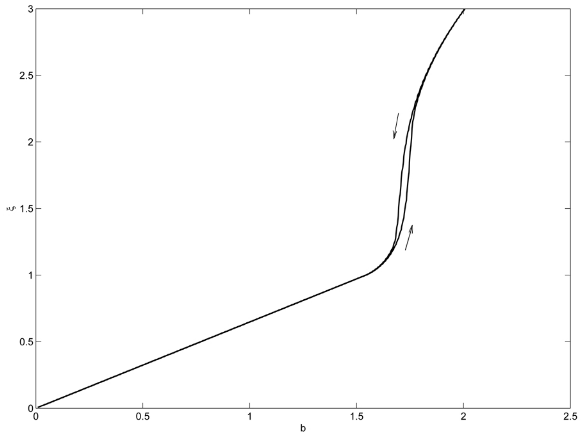


Fig. 1. Amplitude of $-\dot{x} - x + u$ versus the amplitude $b$ of the excitation signal $u(t) = b \sin \omega t$ with $\omega = 0.75$.

So it may be expected that at least for small amplitude and frequency of the excitation signal, the corresponding periodic steady state solution is globally asymptotically stable as well. As for relatively large amplitudes, a simple numerical experiment illustrates their effect, as is depicted in Fig. 1. Specifically, let us increase the amplitude $b$ of the excitation from zero so slowly that the transient behavior becomes negligible. In doing so, we see that the amplitude of the argument $-\dot{x} - x + u$ of the saturation nonlinearity increases gradually until a jump occurs. Surprisingly, if one then starts to slowly decrease $b$, the amplitude of $\xi$ demonstrates a hysteresis-like behavior clearly displayed in Fig. 1 for $\omega = 0.75$.

It follows that the same input signal gives rise to multiple steady-state periodic solutions for $b$'s beneath the hysteresis loop (approximately for $b \in (1.6, 1.8)$), which is incompatible with global stability of any of them. With practical relevance in mind, it would be interesting to obtain conditions on the input signal that guarantee global stability of the steady-state forced oscillations in systems with saturation in feedback.

It is not surprising that the system at hand fails to satisfy the incremental version of the circle criterion since this criterion guarantees input-independent stability of forced oscillations. This system also does not satisfy the input-dependent criterion based on Zames-Falb integral quadratic constraints and elaborated in [14]. Partly in view of these, we will get rid of quadratic Lyapunov functions in favor of special non-quadratic functions by developing some ideas recently set forth in [13].

## III. Convergent systems

In this paper, we consider systems of the form

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \tag{1}$$

with the input $u = u(t)$ taken from a given class $\mathcal{U}$ of admissible input functions $u(\cdot)$. The right-hand side $f(\cdot)$ of (1) satisfies mild regularity assumptions that guarantee existence and uniqueness of solution $x(\cdot)$ with $x(t_0) = x_0$ for any input $u(\cdot) \in \mathcal{U}$ and initial data $t_0, x_0$. This solution defined on the maximal interval is denoted by $x(t, t_0, x_0, u(\cdot))$, where the arguments from the set $t_0, x_0, u(\cdot)$ that are clear from the context will be typically dropped for the sake of brevity. In this section, we introduce the notion of convergent systems, which extends the definition given by Demidovich [2], [3].

*Definition 3.1:* The system (1) is said to be *uniformly convergent* for the input $u(\cdot) \in \mathcal{U}$ if there exists a solution $\bar{x}_u(t)$ of (1) satisfying the following conditions:

i) $\bar{x}_u(t)$ is defined and bounded on the entire time axis $(-\infty, +\infty)$;

ii) $\bar{x}_u(t)$ is uniformly globally asymptotically stable.

Any $\bar{x}_u(t)$ satisfying i) and ii) is called the *steady-state solution*.

Even if the " uniformity" requirement is dropped in ii), any solution " forgets" its initial data by converging to a steady-state solution, which is independent of these data. However in this case, the steady-state solution may be non-unique though the difference between such solutions annihilates as time progresses: $||x_1(t) - x_2(t)|| \to 0$ as $t \to \infty$ for any steady-state solutions $x_1(t)$ and $x_2(t)$. Conversely for **uniformly** convergent systems, the steady-state solution is unique and moreover, this is the only solution that is bounded on the entire time axis $(-\infty, +\infty)$ [15].

In the original definition of convergent systems given in [2], the steady-state solution is required to be unique and merely globally asymptotically stable, not necessarily uniformly. Definition 3.1 does not contain a request for uniqueness since this property follows from ii) [15].

Our interest is focused on the situation, rather ubiquitous in applications, where the system is not uniformly convergent for all feasible inputs $u(\cdot) \in \mathcal{U}$ though it possesses this property within some non-empty proper subset $\mathcal{U}_{\text{conv}} \subset \mathcal{U}$, like in the example from Sect. II. The objective is to obtain an explicit estimate of this subset. A criterion for uniform convergence providing such estimate in this situation is said to be *input-dependent*.

## IV. Stability analysis via averaging functions

The objective of this section is to highlight the major ideas underlying the proposed techniques of the convergence analysis. To this end, we systematically sweep away technical details, focus on main points, and simplify the matters by assuming the right-hand side of (1) smooth in both arguments, which in fact does not affect the above ideas.

In the most general outline, the following three steps can be carried out for $u(\cdot) \in \mathcal{U}$ to establish that the system is uniformly convergent:

**s1)** Proof that the system is uniformly globally bounded in the sense given by the definition from (Sec. 4.8 in[16]);

**s2)** Proof that there exists a solution that is defined and bounded on the entire time axis;

**s3)** Proof that this solution is uniformly globally asymptotically stable.

The first two steps are preliminary and can be carried out by more or less standard methods. The property from s1) is not superfluous since it is evidently inherent in any uniformly convergent system; this property can be established e.g., by means of a proper Lyapunov-like function. Based on s1), the property from s2) can be directly derived by the standard arguments. These preliminary steps not only prove i) from Definition 3.1 but also permit us to shrink the zone of attention from the entire unbounded state space $\mathbb{R}^n$ into its bounded subset that absorbs all solutions and exists since the system is uniformly globally bounded.

Main troubles are encountered in step s3), where we in fact should compare two solutions $x_1(t) := x(t, t_0, x_0), x_2(t) := x(t, t_0, y_0)$ and establish their convergence to each other: $d(t) := \|x_2(t) - x_1(t)\| \to 0$ as $t \to \infty$. Discussion of this step is started with the simple observation: by the multi-dimensional variant of the Lagrange's mean value theorem,

$$d(t) \leq \sup_{0 \leq \theta \leq 1} \left\| \frac{\partial x}{\partial x_0}(t, t_0, x_0 + \theta(y_0 - x_0)) \right\| d(0).$$

Another useful and well-known fact is that here

$$X(t, t_0, x_0) := \frac{\partial x}{\partial x_0}(t, t_0, x_0)$$

is the fundamental solution to the following linear time-varying system

$$\dot{\xi} = J(t)\xi, \quad \text{where} \quad J(t) := \frac{\partial f}{\partial x}\big[x(t, t_0, x_0), u(t)\big]. \quad (2)$$

Combining these observations shows that step s3) can be boiled down to analysis of the parametric variety of linear differential equations (2) with the parameters $t_0, x_0, u(\cdot)$. Specifically, it suffices to prove that for some $\mathcal{K}$-function $\kappa(\cdot)$ [16] and $\varepsilon > 0$, the following implication holds

$$||x_0|| \leq a \Rightarrow ||X(t, t_0, x_0)|| \leq \kappa(a) e^{-\varepsilon(t-t_0)} \; \forall t \geq t_0. \quad (3)$$

A standard way to establish estimates like this is via the use of Lyapunov functions $V(\xi)$ for the parametric variety of linear systems (2). The simplest approach is to try a parameter-independent quadratic function $V(\xi) = \xi^\top P \xi$ with a positive definite matrix $P$ for the role of the Lyapunov function common for all parameters, including $u(\cdot)$'s. By following these lines, one may derive stability conditions similar to those due to Demidovich, which may also be written in the forms of frequency domain inequalities for Lur'e systems; see [10] for details. The main drawback of this approach is that it is, on the one hand, too restrictive by reliance on existence of a common quadratic Lyapunov function and, on the other hand, is too rough in account for the effect of the input. As a result, its typical outcome was an input-independent criterion up to now.

A more sophisticated and flexible approach can be borrowed from [17], where the notion of the Finsler-Lyapunov function was introduced for contractivity analysis of dynamical systems on smooth manifolds. By treating $\mathbb{R}^n$ as a smooth manifold and observing that $\xi$ in (2) has the natural sense of the tangent vector, we arrive at the idea to enlarge the class of candidate Lyapunov functions by allowing the matrix $P$ to depend on the state and time: $V(\xi, t) = \xi^\top P[x(t), t]\xi$. This potentially provides better account for the effect of the input $u(\cdot)$ since the employed function depends on $u(\cdot)$ via the relation $x(t) = x[t, t_0, x_0, u(\cdot)]$. However practically this class is too wide to elaborate verifiable criteria to judge whether it does contain a Lyapunov function.

To cope with this trouble, we restrict the class of candidate Lyapunov functions based on the concept of the "averaging function" from the dimension theory in dynamical systems [18], [19]. The overall effect from this may be interpreted as the use of quadratic functions with simultaneously weakening the conventional requirement to the Lyapunov functions: instead of decaying at any time instant, decaying only "on average" is requested; for a similar approach, see also [20]. In the remainder of the section, we outline a particular way to perform this, thus describing the main methodological contribution of the paper.

### A. *Averaging functions*

To the best of the authors knowledge, the concept close to averaging functions appeared first in studies related to dimension theory for systems of differential and difference equations (see e.g., [18], [19] and literature therein). The concerned ideas and terminology were developed and adapted to stability analysis in [13].

Specifically, we limit attention to the Lyapunov function candidates of the following form:

$$V(\xi, x, t) = \xi^\top P \xi \exp[w(x, t)], \quad (4)$$

where $P$ is a positive definite matrix and $w(x, t)$ is a scalar differentiable function of state $x$ and time $t$, called the *averaging function*. Formula (4) indeed gives a Lyapunov function if the matrix

$$J(t)^\top P + PJ(t) + \dot{w}P \quad (5)$$

is uniformly negative definite and $w[x(t), t]$ is a bounded function of time. In this case, the quadratic form $\mathcal{V}(\xi) := \xi^\top P \xi$ is also a Lyapunov function if $w$ is constant or merely non-decaying $\dot{w} \geq 0$. However most interesting applications of the functions (4) employ $w$ with non-sign-definite derivative $\dot{w}$ for which $\mathcal{V}(\xi)$ is not a Lyapunov function in general. However the uniform negativity of the matrix (5) easily implies that for some $\varepsilon > 0$,

$$\begin{aligned} \mathcal{V}[\xi(t)] &\leq \mathcal{V}[\xi(t_0)] e^{-\varepsilon(t-t_0) - \int_{t_0}^t \dot{w}(s) ds} \\ &= \mathcal{V}[\xi(t_0)] e^{-\varepsilon(t-t_0) - \{w[x(t),t] - w[x(t_0),t_0]\}}. \quad (6) \end{aligned}$$

Since here $w[x(t), t]$ is bounded, the first summand in the sum $\varepsilon(t - t_0) + \{w[x(t), t] - w[x(t_0), t_0]\}$ dominates over the second one as $t \to \infty$, which means the exponential convergence and entails the required estimate of the from (3). However (6) does not imply that $\mathcal{V}[\xi(t)]$ decays monotonically, like Lyapunov functions do, since the second summand may be negative and dominate over the first one at any time $t$. The decay results from averaging the effects of $w$ over large enough time intervals. This explains both terminology and the main benefit: a possible conservatism caused by quadratic Lyapunov functions is reduced by allowing the function to decay not monotonically but only “on average” over extended time horizons.

Rigorous criteria for stability of forced solutions via the averaging functions are offered in [13].

In this paper, we equip the general approach from [13] by a novel technique of finding averaging functions. The idea is to utilize input-output properties of the system (1) with properly chosen output and the associated dissipation inequalities. In doing so, we employ the averaging technique due to Steklov (see, e.g. [21]), which is briefly outlined in the next subsection. It will be used to sharpen the relevant upper estimate of $\dot{w}$.

### *B. Steklov's averaging technique*

This technique is used in various areas, providing area-dependent benefits. For example, it sometimes reduces, more or less, stability analysis of linear time-varying differential equations to that of an LTI system via the so-called $H$-transformation [22], [21]. This is of especial interest for this paper, which deals with time-varying inputs though in the context of non-linear systems.

The discussed technique is based on replacement of a concerned function $r(\cdot)$ by its *Steklov average* with a certain step $T$:

$$r^T(t) = \frac{1}{T}\int_t^{t+T} r(\tau)d\tau.$$

In doing so, various other relations between the original and averaged functions appear to be useful. Now we present some of them, which are of particular interest for this paper. Specifically, they will be used to transform input-output dissipation inequalities into a tractable form via averaging oscillating inputs and related terms.

The symbol $L_2^{\text{loc}}[\mathbb{R} \to \mathbb{R}]$ stands for the space of all measurable functions $v(\cdot): \mathbb{R} \to \mathbb{R}$ that are square integrable on any bounded interval; evidently, $L_\infty[\mathbb{R} \to \mathbb{R}] \subset L_2^{\text{loc}}[\mathbb{R} \to \mathbb{R}]$. Given $v(\cdot) \in L_2^{\text{loc}}[\mathbb{R} \to \mathbb{R}]$ and $t_0 \in \mathbb{R}$, we introduce the following function of $t \in \mathbb{R}$ and $T > 0$, which is the (non-normalized) bias caused by Steklov averaging of $v^2(\cdot)$

$$\sigma(t,T) := \int_{t_0}^{t} \left( \frac{1}{T}\int_{\tau}^{\tau+T} v^2(s)ds - v^2(\tau) \right) d\tau. \quad (7)$$

The first relation of interest is given by the following lemma.

*Lemma 4.1:* $\sup_t |\sigma(t,T)| \le T \cdot \operatorname{ess\,sup}_t v^2(t)$, where ess sup is the essential supremum.[1]

*Proof:* See Lemma 5.4.1 in [21] (see eq. (5.4.4)), see also [22], (p. 541). ■

To state the next relation, we need the following.

*Definition 4.2:* For $v(\cdot) \in L_2^{\text{loc}}\big[\mathbb{R} \to \mathbb{R}\big]$, the *uniform root mean square (URMS) value* is the quantity

$$\mathbf{URMS}(v) := \limsup_{T\to\infty} \sup_{t_0} \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} v^2(\tau)d\tau}.$$

[1] In other words, this is the least upper bound that is valid everywhere, except on a set of measure zero.

In other words, this is the asymptotical (as the step $T \to \infty$) value of the least upper bound on the square root of the Steklov average of $v^2(\cdot)$. Simultaneously, this is an estimate of the mean power of the signal $v(\cdot)$. Let $\mathcal{B}_\beta (\beta \ge 0)$ stand for the set of all $v(\cdot) \in L_2^{\text{loc}}\big[\mathbb{R} \to \mathbb{R}\big]$ such that $\mathbf{URMS}(v) \le \beta$.

*Lemma 4.3:* Suppose that $v(\cdot) \in \mathcal{B}_\beta$. For any $\varepsilon > 0$, there is a positive number $T > 0$ such that for almost all $t$,

$$\dot{\sigma}(t,T) := \frac{\partial \sigma}{\partial t}(t,T) \le \beta^2 - v^2(t) + \varepsilon.$$

This holds irrespective of the choice of $t_0$ in (7).

The ideas outlined in this and previous subsections underly a self-contained technique sufficient to derive constructive input-dependent criteria for uniform convergence.

## V. Input-dependent criteria for convergence of systems with saturation in feedback

We consider the following system

$$\dot{x} = Ax + B\,\mathrm{sat}(Fx+u), \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R} \qquad (8)$$

with the constant matrices $A, B, F$ of appropriate dimensions. In (8), $\mathrm{sat}(\cdot)$ is the standard saturation function $\mathrm{sat}\, z = \mathrm{sign}\, z \min\{1, |z|\}$ and $u = u(t)$ is the excitation input. It is generated by the following linear pre-filter

$$\begin{array}{rcl} \dot{\zeta} &=& E\zeta + Gv, \qquad \zeta \in \mathbb{R}^m, \quad v \in \mathbb{R} \\ u &=& H\zeta + Qv. \end{array} \qquad (9)$$

The mean power of the filter input $v(\cdot)$ is upper limited by a known constant $\beta > 0$. More precisely, we impose the following.

*Assumption 5.1:* The input $u(t)$ to system (8) is a bounded on $(-\infty, +\infty)$ output of the auxiliary system (9) with $v(\cdot) \in \mathcal{B}_\beta$.

The pre-filter (9) is introduced to model, if necessary, the spectral properties of $u(\cdot)$; in some situations, one can take $u = v$ (i.e., $H = 0, Q = 1$) and (9) can be ignored.

We also assume that not only the mean power of the signal $v(\cdot)$ but also its influence on the system (8) can be estimated. Specifically, we impose the following.

*Assumption 5.2:* The $L_2$-gain from $v$ to $s := \mathrm{sat}(Fx+u)$ does not exceed a constant $\gamma$: there exists a positive definite, radially unbounded storage function $W(x,\zeta)$ that satisfies the following dissipation inequality:

$$\dot{W} \le -\,\mathrm{sat}^2(Fx+u) + \gamma^2 v^2. \qquad (10)$$

Before proceeding to criteria for convergence of the system (8), it should be noted that the discussion from Sect. IV is not immediately applicable in full to this system since the right-hand side of (8) is not smooth, as was assumed in that section. At the same time, its major points remain true modulo standard technical complements.

The convergence criterion is based upon the following assumption.

*Assumption 5.3:* There exist a positive definite matrix $P = P^\top$ and numbers $\lambda_1 > 0$, $\lambda_2$ such that the following matrix inequalities hold

$$\begin{array}{rcl} (A+BF)^\top P + P(A+BF) &\le& -\lambda_1 P, \\ A^\top P + PA &\le& \lambda_2 P. \end{array}$$

For conciseness of presentation, we shall focus on the most troublesome case where $\lambda_2 > 0$. The point is that the case $\lambda_2 < 0$ is of little interest, since in this case the system (8) is quadratically convergent and there are no restrictions on the input $u$ for which the existence of required $\bar{x}(\cdot)$ with its stability follows. The case $\lambda_2 = 0$ corresponds to marginal stability of the matrix $A$ and in this case the problem has a solution, e.g. for periodic inputs $u$, as follows from the generalization of LaSalle invariance principle for time-varying systems. So, in the sequel, we assume that $\lambda_2$ is positive. In this case the assumption can be satisfied even for hyperbolically unstable $A$, however Assumption 5.2 restricts the class of admissible $A$'s to marginally unstable matrices.

The convergence criterion is established by the following result.

*Theorem 5.4:* Consider system (8) with the admissible input $u(\cdot)$ under Assumptions 5.1, 5.2, 5.3. Suppose the following inequality holds

$$\beta^2\gamma^2 < \frac{\lambda_1}{\lambda_1+\lambda_2}. \tag{11}$$

Then system (8) is uniformly convergent.

## VI. Example: double integrator with saturated linear feedback

Reconsider the example from Sec. II written in the following form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \mathrm{sat}(-x-y+u), \end{aligned} \tag{12}$$

where $u$ is the output of the following linear system

$$\dot{u} = -\varepsilon u + v(t), \quad \varepsilon > 0. \tag{13}$$

System (12) has the form (8) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad F = (-1, \quad -1).$$

To verify Assumption 5.3 one can take the following matrix $P$

$$P = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}.$$

Direct calculations give

$$(A+BF)^\top P + P(A+BF) = -P.$$

Therefore, one can take $\lambda_1 = 1$ in Assumption 5.3. To estimate $\lambda_2$, one has to find the smallest $\lambda$ so that

$$\left(A - \frac{\lambda}{2}I\right)^\top P + P\left(A - \frac{\lambda}{2}I\right) \leq 0.$$

Simple calculations give $\lambda_2 = 2/\sqrt{3}$, or

$$\frac{\lambda_1}{\lambda_1+\lambda_2} = 2\sqrt{3}-3.$$

To estimate the $L_2$-gain from $v$ to $\mathrm{sat}\, z$, $z = -x-y+u$, as required by Assumption 5.2, take the following storage function

$$W(x,y,u) = y^2 + 2\int_0^z \mathrm{sat}\, s ds + \varepsilon u^2$$

Differentiating $W$ with respect to time along with (12,13) yields

$$\begin{aligned} \dot{W} &= -2\,\mathrm{sat}^2 z + 2(-\varepsilon u + v)\,\mathrm{sat}\, z - 2\varepsilon^2 u^2 + 2\varepsilon uv \\ &\leq -\mathrm{sat}^2 z + (-\varepsilon u + v)^2 - 2\varepsilon^2 u^2 + 2\varepsilon uv \\ &\leq -\mathrm{sat}^2 z + v^2 - \varepsilon^2 u^2 \leq -\mathrm{sat}^2 z + v^2 \end{aligned}$$

and hence, $\gamma$ can be taken as 1. Therefore, by Theorem 5.4 system (12,13) is uniformly convergent as soon as $\mathbf{URMS}(v) < 2\sqrt{3}-3$. Taking $\varepsilon$ as small as we wish, it can be concluded from Theorem 5.4 that the system (12) is uniformly convergent for all inputs $u$ that satisfy

$$\mathbf{URMS}(\dot{u}) < 2\sqrt{3}-3.$$

In a particular case of harmonic input $u = b\sin\omega t$ the previous inequality gives the following sufficient condition for uniform convergence:

$$b\omega < \sqrt{4\sqrt{3}-6} \approx 0.9634. \tag{14}$$

It is worth noting that the conditions imposed in Theorem 5.4 do not require knowledge of the bounds of $|\dot{u}|$. If one knows those bounds, like in case of harmonic input, the stability criterion (14) can be sharpened. To illustrate this idea, consider the following function

$$W(x,y,t) = y^2/2 + \int_0^z \mathrm{sat}\, s ds.$$

Differentiating this function with respect to time yields

$$\dot{W} = -\mathrm{sat}^2 z + \dot{u}\,\mathrm{sat}\, z = -\left(\mathrm{sat}\, z + \frac{\dot{u}}{2}\right)^2 + \frac{\dot{u}^2}{4}.$$

If for all $t$ $|\dot{u}(t)| < 2$, the first term in the last expression is strictly negative in the saturation mode. This fact can be exploited similar to Example in [13]. Simple calculations with the same $P$ as above along the lines of Example in [13] give the following stability criterion for the harmonic input:

$$b\omega < 2\sqrt{6+4\sqrt{3}}\left(2+\sqrt{3}\right) - 4\left(2\sqrt{3}+3\right) \approx 0.9814. \tag{15}$$

Clearly, this criterion is better than (14). However the new criterion can be applied to a much wider class of inputs. To illustrate how conservative the criterion derived, we performed computer simulation similar to the experiment described in Sec. II. The results of computer simulations are depicted on Figure 2. The area below the smooth curve selects the parameters $b, \omega$ for which the system is convergent according to (14). The dots indicate the points for which computer simulation reveals multiple periodic solutions, which means that around those points the system is not convergent. It is worth mentioning that to locate a point on figures similar to Fig. 1 that corresponds to multiple periodic solutions is not a trivial problem: the angle at which two branches of the hysteresis curve diverge can be very acute. Therefore to find the onset of convergence via computer simulation will require a more advanced numerical technique than that described in Sec. II.

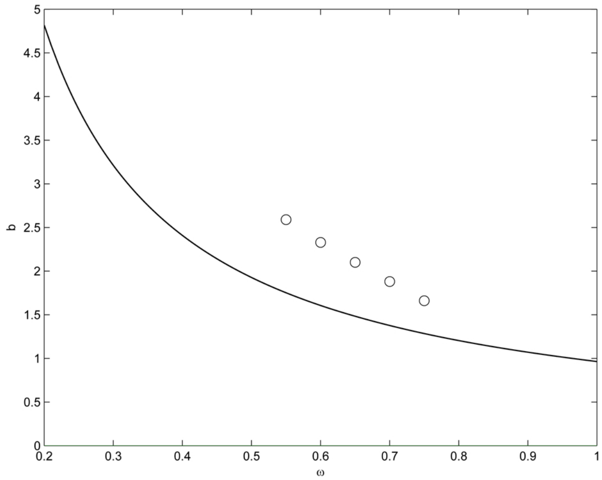


Fig. 2. The area below the smooth curve selects the region in parameter space that guarantees convergence. The dots indicate the points where numerically it was found that the system is not convergent.

## VII. Conclusions

By a technique based on the so-called averaging functions, a novel constructive criterion is established that allows to prove stability of forced oscillations in the case when the classical incremental circle criterion fails. For linear systems with saturation in feedback and external input it was demonstrated that one can benefit from the averaging functions in the special form: the averaging function is the sum of the storage function that estimates the $L_2$-gain from the input to the output of the saturation nonlinearity plus an input-dependent functional. Development of numerical tools, based on frequency domain inequalities and/or Linear Matrix Inequalities, that support the proposed technique, is the topic of ongoing research.

## References


[1] V. Pliss, *Nonlocal Problems of the Theory of Oscillations*. London: Academic Press, 1966.

[2] B. Demidovich, *Lectures on Stability Theory*. Moscow: Nauka, 1967, in Russian.

[3] A. Pavlov, A. Pogromsky, N. van de Wouw, and H. Nijmeijer, "Convergent dynamics, a tribute to Boris Pavlovich Demidovich," *Systems and Control Letters*, vol. 52, pp. 257–261, 2004.

[4] J. LaSalle and S. Lefschetz, *Stability by Liapunov's Direct Method with Applications*. New York: Academic Press, 1961.

[5] T. Yoshizawa, *Stability Theory by Liapunov's Second Method*. Tokyo: The Mathematical Society of Japan, 1966.

[6] W. Lohmiller and J. Slotine, "On contraction analysis for nonlinear systems," *Automatica*, vol. 34, pp. 683–696, 1998.

[7] V. Fromion, S. Monaco, and D. Normand-Cyrot, "Asymptotic properties of incrementally stable systems," *IEEE Trans. Automatic Control*, vol. 41, pp. 721–723, 1996.

[8] D. Angeli, "A Lyapunov approach to incremental stability properties," *IEEE Transactions on Automatic Control*, vol. 47, pp. 410–421, 2002.

[9] B. S. Rüffer, N. van de Wouw, and M. Mueller, "Convergent systems vs. incremental stability," *Systems & Control Letters*, vol. 62, pp. 277—285, 2013.

[10] V. A. Yakubovich, "The matrix-inequality method in the theory of the stability of nonlinear control systems. I: The absolute stability of forced vibrations," *Automation and Remote Control*, vol. 7, pp. 905–917, 1965.

[11] Y. N. Bibikov, "Convergence in Lienard equation with a force term," *Vestnik Leningradskogo Universiteta*, vol. 7, pp. 13–15, 1976.

[12] G. Leonov, "A frequency criterion of stabilization of nonlinear systems by a harmonic external action," *Avtomatika i Telemekhanika*, vol. 1, pp. 169–174, 1986.

[13] A. Pogromsky and A. Matveev, "A non-quadratic criterion for stability of forced oscillations," *Systems and Control Letters*, vol. 62, no. 5, pp. 408–412, 2013.

[14] A. Rantzer and A. Megretski, "Harmonic analysis of nonlinear and uncertain systems," in *Proc. American Control Conference*, Philadelphia, Pensilvania, 1998, pp. 3654–3658.

[15] A. Pavlov, N. van de Wouw, and H. Nijmeijer, *Uniform Output Regulation of Nonlinear Systems: A Convergent Dynamics Approach*. Boston: Birkhauser, 2005.

[16] H. Khalil, *Nonlinear Systems*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 1996.

[17] F. Forni and R. Sepulchre, "A differential Lyapunov framework for contraction analysis," *arXiv:1208.2943v1 [cs.SY]*, 2012.

[18] V. A. Boichenko, G. A. Leonov, and V. Reitman, *Dimension Theory for Ordinary Differential Equations*. Wiesbaden, Germany: Teubner Verlag, 2005.

[19] G. A. Leonov, *Strange Attractors and Classical Stability Theory*. St. Petersburg, Russia: St. Petersburg University Press, 2008.

[20] D. Aeyels and J. Peuteman, "A New Asymptotic Stability Criterion for Nonlinear Time-Variant Differential Equations," *IEEE Transactions on Automatic Control*, vol. 43, no. 7, pp. 968–971, 1998.

[21] L. Adrianova, *Introduction to Linear Systems of Differential Equations*. Rhode Island: American Mathematical Society, Providence, 1995.

[22] B. Bylov, R. Vinograd, D. Grobman, and V. Nemyckii, *The theory of Lyapunov exponents and its applications to problems of stability*. Moscow: Nauka, 1966, in Russian.