Separable Lyapunov functions for monotone systems

Anders Rantzer[1], Björn S. Rüffer[2] and Gunther Dirr[3]

***Abstract*— Separable Lyapunov functions play vital roles, for example, in stability analysis of large-scale systems. A Lyapunov function is called max-separable if it can be decomposed into a maximum of functions with one-dimensional arguments. Similarly, it is called sum-separable if it is a sum of such functions. In this paper it is shown that for a monotone system on a compact state space, asymptotic stability implies existence of a max-separable Lyapunov function. We also construct two systems on a non-compact state space, for which a max-separable Lyapunov function does not exist. One of them has a sum-separable Lyapunov function. The other does not.**

I. Introduction

A system of differential equations is called monotone if a partial order relationship between initial conditions is preserved by the dynamics. For systems on $\mathbb{R}^n_+ = [0,\infty)^n$ that are monotone with respect to the component-wise partial order on $\mathbb{R}^n$, two types of Lyapunov functions have been of interest in the recent literature on the stability analysis of large-scale interconnected nonlinear systems. These are the *max-separable* Lyapunov function

$$V(x) = \max_{i=1,\dots,n} V_i(x_i), \tag{1}$$

e.g., in [1] and the *sum-separable* Lyapunov function

$$V(x) = \sum_{i=1}^{n} V_i(x_i), \tag{2}$$

e.g., in [2], [3], [7]. The Lyapunov functions (1) and (2) are also of recent interest in decentralized control, see [4], [5].

Roughly speaking and by way of an example, separable Lyapunov functions appear in the construction of Lyapunov functions for composite systems. In applications such a composite system appears as an interconnection of many stable subsystem. There it is usually assumed that every such subsystem is endowed with a suitable Lyapunov function that quantifies the subsystem's stability with respect to input from other subsystems. More precisely, one could assume that every subsystem is input-to-state stable (ISS) with an ISS Lyapunov function V_i. For this case it was shown, e.g., in [1], that under suitable conditions $V(x) = \max_i \sigma_i\big(V_i(x_i)\big)$ is a Lyapunov function for the composite system, where the functions σ_i are appropriate scaling functions. Clearly, this composite Lyapunov function is of the form (1).

However, when subsystems are allowed to satisfy relaxed stability assumptions, e.g, they are only assumed to be *integral* input-to-state stable (iISS), then it was found that the same construction from [1] does not necessarily work. Instead, a construction based on (2) has been used successfully at different occasions.

Both of these constructions are related to monotone systems, as the respective stability conditions for large-scale systems can always be translated into a stability condition on a lower-dimensional, monotone comparison system [6], [7].

A natural question thus is: If Lyapunov functions of the form (2) can seem to handle "more general" types of interconnections of stable subsystems, is the set of monotone (comparison) systems admitting such a Lyapunov function bigger than the class of systems only admitting a Lyapunov function of the form (1)?

In this work we show that for an asymptotically stable, monotone system on a compact state space there always exists a max-separable Lyapunov function. Furthermore, we show that the compactness assumption is indeed essential for this construction, by giving a simple example of a system with non-compact state-space for which no max-separable Lyapunov function exists. We also construct a system (on a non-compact state-space) that neither has a max-separable nor a sum-separable Lyapunov function.

The paper is organized as follows: First we give precise definitions of what we mean by a monotone system, partial order, asymptotic stability, etc. Then in Section III we present our main result, namely the aforementioned construction of max-separable Lyapunov functions. The counter-examples are given in Section IV.

II. Notation

We consider $\mathbb{R}^n$ equipped with the component-wise partial order, which we denote by $x \le y$ if $x_i \le y_i$ for all i, $x < y$ if $x \le y$ but $x \ne y$, and $x \ll y$ if $x_i < y_i$ for all i. A map $F : \mathbb{R}^n \to \mathbb{R}^n$ is *monotone* if $x \le y$ implies $F(x) \le F(y)$. For a partially ordered set A we denote by $A_+ := \{a \in A : a \ge 0\}$.

In this work we consider systems of the form

$$\dot{x} = f(x) \tag{3}$$

with $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ locally Lipschitz and $f(0) = 0$. This guarantees local existence and uniqueness of solutions. Associated with this system is the flow map $\varphi : \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$, which satisfies $\varphi(t,\varphi(s,x)) = \varphi(t+s,x)$ and $\varphi(0,x) = x$ for all $t,s \in \mathbb{R}_+$ and $x \in \mathbb{R}^n_+$.

[1]A. Rantzer is with Automatic Control LTH, Lund University, Box 118, SE-221 00 Lund, Sweden, rantzer at control.lth.se.

[2]Björn S. Rüffer is with Signal & System Theory Group, EIM-E, Universität Paderborn, Warburger Str. 100, Germany, bjoern@rueffer.info.

[3]Gunther Dirr is with Institut für Mathematik, Universität Würzburg, Campus Hubland Nord, Emil-Fischer-Str. 40, 97074 Würzburg, Germany dirr@mathematik.uni-wuerzburg.de.

Throughout this paper we will assume that system (3) is monotone, i.e., $x \leq y$ implies $\varphi(t,x) \leq \varphi(t,y)$ for all $t \in \mathbb{R}_+$. This holds if and only if f satisfies the Kamke-Müller conditions, cf. [8],

$$x \leq y \text{ and } x_i = y_i \implies f_i(x) \leq f_i(y). \tag{4}$$

Note that existence and uniqueness of solutions dictates that at least for points $x \gg 0$ the flow map $\varphi(t,x)$ can also be evaluated for small negative times.

The origin is *asymptotically stable* if it is attractive and stable in the sense of Lyapunov. It is *globally asymptotically stable* (GAS) if it is asymptotically stable and its region of attraction is the entire $\mathbb{R}^n_+$.

III. Separable Lyapunov functions

Our main result shows that one can always find a max-separable Lyapunov function on compact sets.

Theorem 1: Let (3) be a monotone system so that the origin is globally asymptotically stable. Suppose that the system leaves the compact set $X \subset \mathbb{R}^n_+$ invariant. Then there exist strictly increasing functions $V_k : \mathbb{R}_+ \to \mathbb{R}_+$ for $k = 1,\ldots,n$ such that $V(x) = \max\{V_1(x_1),\ldots,V_n(x_n)\}$ satisfies

$$\frac{d}{dt}V(\varphi(t,x^0)) = -V(\varphi(t,x^0))$$

for all $x^0 \in X$, $x^0 \gg 0$.

Remark 1. If a compact set X is not invariant to begin with, then one can consider instead the invariant set

$$Y := \overline{\bigcup_{t\geq 0} \varphi(t,X)}.$$

Proof. Define $\overline{x}_k := 1 + \sup\{x_k : x \in X\}$. Then, due to monotonicity of the system we have for all $x \in X$ that

$$0 \leq \max_k \varphi_k(t,x) \leq \max_k \varphi_k(t,\overline{x}) \longrightarrow 0 \quad \text{ as } t \to \infty$$

where $\varphi_k(t,\overline{x})$ denotes the kth component of $\varphi(t,\overline{x})$. For $x \in X$ define

$$\begin{aligned} T_k(x_k) &:= \max\left\{\tau : x_k \leq \varphi_k(t,\overline{x}) \text{ for all } t \in [0,\tau]\right\} \\ T(x) &:= \max\left\{\tau : x \leq \varphi(t,\overline{x}) \text{ for all } t \in [0,\tau]\right\} \end{aligned}$$

where x_k and $\varphi_k(t,\overline{x})$ denote the kth components of x and $\varphi(t,\overline{x})$. Then $T(x) = \min\{T_1(x_1),\ldots,T_n(x_n)\}$. It follows from compactness of X and global asymptotic stability of $x = 0$ that $T(x)$ is finite for all $x \in X$ with $x \neq 0$. Moreover

$$\begin{aligned} &T(\varphi(\epsilon,x)) \\ &= \max\left\{\tau : \varphi(\epsilon,x) \leq \varphi(t,\overline{x}) \text{ for all } 0 \leq t \leq \tau\right\} \\ &= \max\left\{\tau : \varphi(\epsilon,x) \leq \varphi(t,\overline{x}) \text{ for all } \epsilon \leq t \leq \tau\right\} \\ &= \max\left\{\tau : \varphi(\epsilon,x) \leq \varphi(t+\epsilon,\overline{x}) \text{ for all } 0 \leq t \leq \tau-\epsilon\right\} \\ &\geq \max\left\{\tau : x \leq \varphi(t,\overline{x}) \text{ for all } 0 \leq t \leq \tau-\epsilon\right\} \\ &= \epsilon + T(x) \end{aligned}$$

The inequality is due to monotonicity of the dynamics. This shows that the map $t \mapsto T(\varphi(t,x))$ is a strictly increasing function of t. We will prove the desired properties for the functions

$$V_k(z) := e^{-T_k(z)}, \qquad k = 1,\ldots,n$$

where $k = 1,\ldots,n$. First notice that V_k is strictly decreasing due to the definition of T_k. Define ϵ such that $0 < \epsilon < T(x)$ for all $x \in X$. With

$$V(x) := \max\left\{V_1(x_1),\ldots,V_n(x_n)\right\} = e^{-T(x)}$$

it follows for $x \in X$ that It follows that

$$\begin{aligned} \frac{d}{dt}V(\varphi(t,x))\Big|_{t=0} &= -e^{-T(\varphi(t,x))}\frac{d}{dt}T(\varphi(t,x))\Big|_{t=0} \\ &\leq -e^{-T(\varphi(t,x))}\Big|_{t=0} = -V(x). \end{aligned}$$

We note that V is by construction positive and strictly decreasing along trajectories. This completes the proof. □

The reasoning of the previous theorem does not work for an arbitrary monotone system with globally asymptotically stable origin, as we will see in the examples of Section IV-A. However, the following result holds.

Corollary 1: Let (3) be a monotone system so that the origin globally asymptotically stable. Suppose the there is a trajectory $\overline{\varphi}(t) \in \mathbb{R}^n_+$ such that

- $\overline{\varphi}(t)$ is defined for all forward and backward times;
- $\lim_{t\to\infty} \overline{\varphi}(t) = 0$ and $\lim_{t\to-\infty} \overline{\varphi}_k(t) = \infty$ for all k.

Then there exists a max-separable Lyapunov function.

Proof. The proof is essentially the same as the construction given in the proof of Theorem 1. First we let

$$\begin{aligned} T_k(x_k) &:= \max\{\tau : \overline{x}_k \leq \overline{\varphi}_k(t) \text{ for all } t \in [0,\tau]\} \\ T(x) &:= \max\{\tau : x \leq \overline{\varphi}(t) \text{ for all } t \in [0,\tau]\} \end{aligned}$$

for $k = 1,\ldots,n$. Again $T(x) = \min_k T_k(x_k)$ and we define

$$V(x) := e^{-T(x)} = \max_k e^{-T_k(x_k)} =: \max_k V_k(x_k).$$

Observe that $V(x) \to \infty$ as $\|x\| \to \infty$. The remainder of the proof is the same as for Theorem 1. □

IV. Examples

The following two examples demonstrate that compactness of the state-space is indeed crucial for the existence of separable Lyapunov functions. In both cases the origin is globally asymptotically stable on $\mathbb{R}^n_+$.

A. A system with a sum-separable Lyapunov function that does not exhibit a max-separable Lyapunov function

Consider the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{x}{1+x} + y \\ -y \end{pmatrix} =: f(x,y). \tag{5}$$

The right-hand side is locally Lipschitz continuous, satisfies $f(0,0) = 0$, as well as the Kamke-Müller conditions (4).

Hence (5) defines a monotone system on $\mathbb{R}^2_+$. Figure 1 shows how the state space is divided into two regions,

$$R_{\texttt{upper}} = \left\{ x \in \mathbb{R}^2_+ : x > 0, \quad y > \frac{x}{1+x} \right\}$$
$$R_{\texttt{lower}} = \left\{ x \in \mathbb{R}^2_+ : x > 0, \quad 0 < y < \frac{x}{1+x} \right\},$$

separated by the dashed line. In the upper region trajectories increase in the first component, while they decrease in the second component. Eventually, they enter the lower region, where both components decrease ad infinitum towards the origin. The shown trajectory is representative for all trajectories passing through $R_{\texttt{upper}}$. Clearly, none of them is unbounded in both components in backward-time. Hence, no trajectory as in Corollary 1 can be used to dominate all points in $\mathbb{R}^n_+$ and the construction of that corollary fails.

Next we show, that there is no "other" max-separable Lyapunov function either. By way of contradiction assume that there is a $V(x,y) = \max\{V_1(x), V_2(y)\}$. We may assume that V_i, $i = 1,2$, are of class $\mathcal{K}_\infty$ and are hence differentiable almost everywhere. Consider the sequence $z^n = (n,2)^T$, $n \geq 1$. There must be some $N \geq 1$ such that for all $n \geq N$ we have $V(z^n) = V_1(z^n_1)$. Observe that $V_1'(s) = \frac{d}{ds}V_1(s) \geq 0$ wherever the derivative exists. But now we have, for $n \geq N$ and at points of differentiability z^n, that

$$\dot{V} = V_1'(z^n) f_1(z^n)$$
$$= V_1'(z^n)\left(2 - \frac{n}{1+n}\right) \geq 0.$$

The same argument would work along any other horizontal line above R_{lower}, so we can actually show that $\dot{V} \geq 0$ on a set of positive measure. This, however, contradicts the fact that V is supposed to be Lyapunov function. Hence, this system does not admit any max-separable Lyapunov function.

Now consider the $\mathcal{C}^1$ function $V(x,y) = x + 2y$. On $\mathbb{R}^2_+$ it is positive definite and radially unbounded. The system is globally asymptotically stable. We have $\dot{V} = \dot{x} + 2\dot{y} = -\frac{x}{1+x} - y < 0$ for all $x > 0$ and $y > 0$. So V must be a Lyapunov function, and very clearly it is sum-separable. This establishes that the origin is globally asymptotically stable.

B. A system that does not exhibit a sum-separable nor a max-separable Lyapunov function

Our second example shows that for non-compact state-space a sum-separable Lyapunov function does not need to exist either.

1) Preliminary step: Consider the following two-dimensional (preliminary) system on $\mathbb{R}_+ \times \mathbb{R}_+$:

$$\begin{aligned} \dot{x} &= \tfrac{y^2}{y^2+1} - x =: \widehat{f}(x,y) \\ \dot{y} &= x - \tfrac{2y^2}{y^2+1} =: \widehat{g}(x,y) \end{aligned} \tag{6}$$

Clearly, if $x > \frac{y^2}{y^2+1}$ then $\widehat{f}(x,y) < 0$ and if $x < \frac{2y^2}{y^2+1}$ then $\widehat{g}(x,y) < 0$. Thus, for $\frac{y^2}{y^2+1} < x < \frac{2y^2}{y^2+1}$ one has $\widehat{f}(x,y) < 0$ and $\widehat{g}(x,y) < 0$, as depicted in Figure 2.

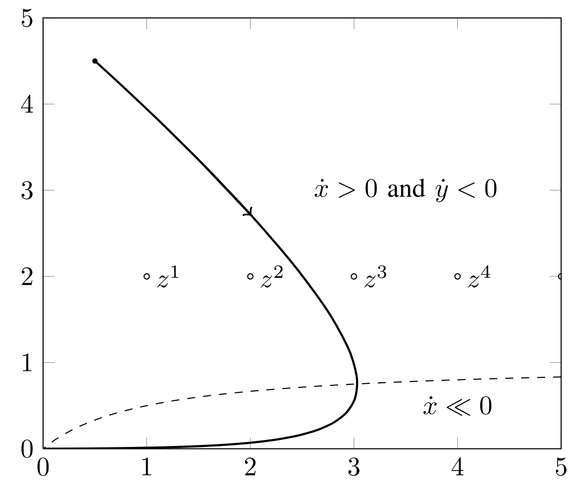

Fig. 1. Sign patterns of the right-hand side of system (5) given in Section IV-A. Although the system is GAS, it does not admit a *global* max-separable Lyapunov function. The simple reason is that no trajectory is unbounded in all components in backward-time.

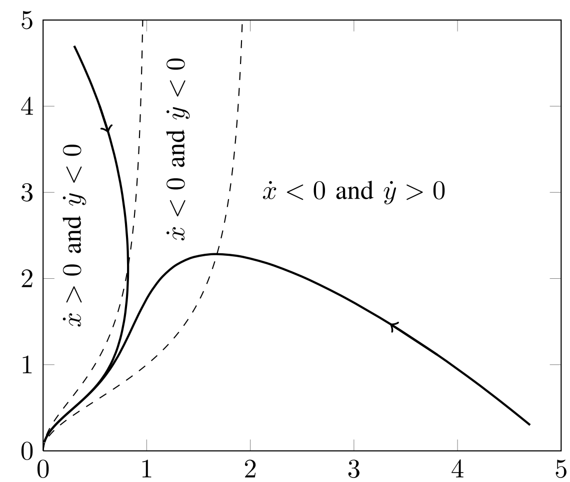

Fig. 2. Sign patterns of the right-hand side of system (6) given in Section IV-B and two representative trajectories. Although the system is GAS, it does not admit a *global* sum-separable Lyapunov function.

Now, assume that for (6) there exists a strict global Lyapunov function of the form

$$L(x,y) = V(x) + U(y), \tag{7}$$

i.e., L is supposed to be differentiable (not necessarily continuously differentiable) on $\mathbb{R}_+ \times \mathbb{R}_+$ and has to satisfy the condition

$$\dot{L}(x,y) := V'(x)\widehat{f}(x,y) + U'(y)\widehat{g}(x,y) < 0 \tag{8}$$

for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(0,0)\}$, where V' and U' denote the ordinary derivative of V and U, respectively.

2) Step 1: Now, we pass from (6) to the following system

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \tag{9}$$

where f equals $\widehat{f}$ and g has the same sign pattern as $\widehat{g}$, yet a different limit behaviour. More precisely, there should exist

$0 < x_* < 1$ and $x^* > 2$ such that

$$\lim_{y\to\infty} g(x_*, y) = 0 \qquad \text{and} \qquad \lim_{y\to\infty} g(x^*, y) = \infty.$$

3) Claim: If there exists a map g with the above properties then (9) does not admit a Lyapunov function of the form (7).

Proof. Assume that (9) has a Lyapunov function of the form (7). This, however, would imply the following (contradictory) limit behaviour for $U'(y)$:

(a) On the one hand, the inequality

$$\dot{L}(x_*, y) = \underbrace{V'(x_*)}_{>0} f(x_*, y) + \underbrace{U'(y)}_{>0} \underbrace{g(x_*, y)}_{<0} < 0$$

implies $\lim_{y\to\infty} U'(y) = \infty$ because $\lim_{y\to\infty} f(x_*, y) = 1 - x_* > 0$ and $\lim_{y\to\infty} g(x_*, y) = 0$.

(b) On the other hand, the inequality

$$\dot{L}(x^*, y) = \underbrace{V'(x^*)}_{>0} \underbrace{f(x^*, y)}_{<0} + \underbrace{U'(y)}_{>0} \underbrace{g(x^*, y)}_{>0} < 0$$

implies $\lim_{y\to\infty} U'(y) = 0$ because $\lim_{y\to\infty} f(x_*, y) = 1 - x^* < 0$ and $\lim_{y\to\infty} g(x_*, y) = \infty$.

□

Thus, once we have shown that such a map g does exist we have also proved that (9) does not admit a Lyapunov function of the form $L = V + U$.

4) Step 2: Here, we explicitly "construct" a map g which satisfies the above requirements. Choose differentiable, positive definite functions $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{y\to\infty} \alpha(y) = 0, \qquad \lim_{y\to\infty} \beta(y) = \infty, \quad \text{and} \quad \lim_{y\to\infty} \alpha(y) \mathrm{e}^{\lambda\beta(y)} = \infty \tag{10}$$

for some suitable $\lambda > 0$. Then, define $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ as follows

$$g(x, y) := \alpha(y)\Big(\mathrm{e}^{\beta(y)\left(x - \frac{2y^2}{y^2-1}\right)} - 1\Big) = \alpha(y)\Big(\mathrm{e}^{\beta(y)\widehat{g}(x,y)} - 1\Big).$$

Obviously, g has the same sign pattern as $\widehat{g}$. Moreover, for $x_* < 2$ and $x^* := 2 + \lambda$ one has the following limit behaviour

$$\lim_{y\to\infty} g(x_*, y) = \lim_{y\to\infty} -\alpha(y) = 0 \quad \text{and}$$

$$\lim_{y\to\infty} g(x^*, y) = \lim_{y\to\infty} \alpha(y) \mathrm{e}^{\beta(y)\left(x^* - 2\right)} = \infty.$$

5) Step 3: Finally, we have to choose α and β such that (9) is monotone and asymptotically stable.

Monotonicity All we have to check is that $\frac{\partial f}{\partial y}$ and $\frac{\partial g}{\partial x}$ are non-negative. Indeed, we find

$$\frac{\partial f}{\partial y}(x, y) = \frac{2y(y^2+1) - 2y^3}{(y^2+1)^2} = \frac{2y}{(y^2+1)^2} \geq 0$$

and

$$\frac{\partial g}{\partial x}(x, y) = \alpha(y)\beta(y)\mathrm{e}^{\beta(y)\left(x - \frac{2y^2}{y^2-1}\right)} > 0.$$

Thus (9) defines a monotone system on $\mathbb{R}_+ \times \mathbb{R}_+$, whenever α and β are strictly positive.

Asymptotic stability First, the forward invariance of $\mathbb{R}_+ \times \mathbb{R}_+$ under the flow of (9) follows straightforwardly by inspection of the vector field (f, g) on the x- and y-axis. Moreover, there are obviously no other equilibria in $\mathbb{R}_+ \times \mathbb{R}_+$ than $(0, 0)$. To prove global asymptotic stability of $(0, 0)$, it suffices to show that all solutions of (9) eventually reach the set

$$\begin{aligned} \Gamma^- :=& \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid f(x, y) < 0 \text{ and } g(x, y) < 0\} \\ =& \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \tfrac{y^2}{y^2+1} < x < \tfrac{2y^2}{y^2+1}\}, \end{aligned}$$

because Γ^- is forward invariant under the flow of (9) and admits $L(x, y) := x + y$ as Lyapunov function. Due to the sign pattern of f and g, the "attractiveness" of Γ^- is easily established once one can guarantee that the vector field (f, g) is complete (no finite escape time). Therefore, one has to choose α and β in a moderate way, e.g.

$$\alpha(y) := \big(\ln(y + c)\big)^{-1} \qquad \text{and} \qquad \beta(y) := \ln\big(\ln(y + c)\big)$$

with $c > \mathrm{e}$. Then clearly the first two limit conditions of (10) are satisfied. Moreover, for any $\lambda > 1$ (and thus for any $x^* > 3$) one has

$$\begin{aligned} \lim_{y\to\infty} \alpha(y)\mathrm{e}^{\lambda\beta(y)} &= \lim_{y\to\infty} \big(\ln(y + c)\big)^{-1} \mathrm{e}^{\lambda \ln\left(\ln(y+c)\right)} \\ &= \lim_{y\to\infty} \big(\ln(y + c)\big)^{\lambda - 1} = \infty. \end{aligned}$$

Now with the above choice of α and β we can prove that Γ^- is "attractive". To this end, define

$$\begin{aligned} \Gamma_f^- &:= \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid f(x, y) < 0\} \\ &= \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \tfrac{y^2}{y^2+1} < x\} \end{aligned}$$

and

$$\begin{aligned} \Gamma_g^- &:= \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid g(x, y) < 0\} \\ &= \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x < \tfrac{2y^2}{y^2+1}\} \end{aligned}$$

Case 1: Let $(x_0, y_0) \in \Gamma_f^- \setminus \Gamma^-$. Then, $\Gamma_f^-(x_0) := \Gamma_f^- \cap \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x \leq x_0\}$ is forward invariant under the flow of (9). This follows easily from the behaviour of the vector field (f, g) on the boundary of $\Gamma_f^-(x_0)$. On $\Gamma_f^-(x_0)$, we can estimate g as follows

$$\begin{aligned} |g(x, y)| &\leq \Big|\alpha(y)\Big(\mathrm{e}^{\beta(y)\left(x - \frac{2y^2}{y^2-1}\right)} - 1\Big)\Big| \\ &\leq \big|\ln(y + c)\big|^x + \big|\ln(y + c)\big|^{-1} \\ &\leq \big|\ln(y + c)\big|^{-1} + \big|\ln(y + c)\big|^{x_0} \\ &\leq \big|\ln(y + c)\big|^{-1} + C(y^{1/x_0})^{x_0} \leq C'y \end{aligned}$$

with appropriate constants $C > 0$ and $C' > 0$. Therefore, finite escape time phenomena can be excluded by a standard Gronwall type estimate. Hence, any solution starting in Γ_f^- has to reach eventually Γ^-.

Case 2: Let $(x_0, y_0) \in \Gamma_g^- \setminus \Gamma^-$. Then, $\Gamma_g^-(y_0) := \Gamma_g^- \cap \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid y \leq y_0\}$ is forward invariant under the flow of (9). Since $\Gamma_g^-(y_0)$ is also bounded the corresponding solution does exist for all $t > 0$.

6) Step 4: Finally, from Figure 2 and the reasoning in Section IV-A it is clear that this system does not have a max-separable Lyapunov function either.

V. Conclusion

This work has shown that globally asymptotically stable monotone systems always have a max-separable Lyapunov function on every compact invariant set. Counter-examples have been provided to show that the compactness assumption is essential.

VI. Acknowledgment

The first author is supported by the LCCC Linnaeus Center and the eLLIIT Excellence Center at Lund University.

References

[1] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM J. Control Optim.*, 48(6):4089–4118, 2010.

[2] Hiroshi Ito. State-dependent scaling problems and stability of interconnected iISS and ISS systems. *IEEE Trans. Autom. Control*, 51(10):1626–1643, 2006.

[3] Hiroshi Ito, Zhong-Ping Jiang, Sergey Dashkovskiy, and Björn S. Rüffer. Robust stability of networks of iISS systems: Construction of sum-type Lyapunov functions. *IEEE Trans. Autom. Control*, 2013. To appear.

[4] Anders Rantzer. Distributed control of positive systems. In *Proc. 50th IEEE Conf. Decis. Control*, pages 6608–6611, Orlando, FL, USA, 2011.

[5] Anders Rantzer. Optimizing positively dominated systems. In *Proc. 51st IEEE Conf. Decis. Control*, pages 272–277, Maui, Hawaii, USA, 2012.

[6] B. S. Rüffer. Small-gain conditions and the comparison principle. *IEEE Trans. Autom. Control*, 55(7):1732–1736, July 2010.

[7] B. S. Rüffer, C. M. Kellett, and S. R. Weller. Connection between cooperative positive systems and integral input-to-state stability of large-scale systems. *Automatica J. IFAC*, 46(6):1019–1027, 2010.

[8] Hal L. Smith. *Monotone dynamical systems*, volume 41 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1995.