Monotone inequalities, dynamical systems, and paths in the positive orthant of Euclidean n-space

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Abstract. Given monotone operators on the positive orthant in n-dimensional Euclidean space, we explore the relation between inequalities involving those operators, and induced monotone dynamical systems. Attractivity of the origin implies stability for these systems, as well as a certain inequality, the no-joint-increase condition. Under the right perspective the converse is also true. In addition we construct an unbounded path in the set where trajectories of the dynamical system decay monotonically, i.e., we solve a positive continuous selection problem.

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1. Introduction

Let $\mathbb{R}_+$ denote the set of nonnegative reals. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}_\infty$, if $f(0) = 0$, and f is continuous, strictly increasing, and unbounded. We write class $\mathcal{G}$ for $\mathcal{K}_\infty \cup \{0\}$, i.e., include the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ which is zero everywhere. To state the problem we define n continuous and monotone functions $\mu_i : \mathbb{R}^n_+ \to \mathbb{R}_+$, satisfying

$$\begin{aligned} \mu_i(x) > 0 \text{ if } x_j > 0 \text{ for all } j = 1, \dots, n, \text{ and} \\ \mu_i(x) < \mu_i(y) \text{ if } x_j < y_j \text{ for all } j = 1, \dots, n\,. \end{aligned} \tag{1.1}$$

In this paper we provide solutions to the following problems:

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Monotone inequalities. Given class $\mathcal{G}$ functions γ_{ij}, $i,j=1,\dots,n$, and continuous functions μ_i, $i=1,\dots,n$, as in (1.1), find, in terms of a given $y=(y_1,\dots,y_n)^T \in \mathbb{R}^n_+$, upper bounds on the set of solutions $x=(x_1,\dots,x_n)^T \in \mathbb{R}^n_+$ of the set of inequalities

$$\begin{aligned} x_1 - \mu_1(\gamma_{11}(x_1),\dots,\gamma_{1n}(x_n)) &\le y_1 \\ &\vdots \\ x_n - \mu_n(\gamma_{n1}(x_1),\dots,\gamma_{nn}(x_n)) &\le y_n\,. \end{aligned} \tag{1.2}$$

Path parametrization. Given class $\mathcal{G}$ functions γ_{ij}, $i,j=1,\dots,n$, and continuous functions μ_i, $i=1,\dots,n$, as in (1.1), find a parametrization $\sigma:\mathbb{R}_+\to\mathbb{R}^n_+$ of a strictly increasing, continuous path starting at the origin, unbounded in all coordinate directions (i.e., $\sigma_i\in\mathcal{K}_\infty$ for all i), and satisfying

$$\begin{aligned} \mu_1(\gamma_{11}(\sigma_1(r)),\dots,\gamma_{1n}(\sigma_n(r))) &< \sigma_1(r) \\ &\vdots \\ \mu_n(\gamma_{n1}(\sigma_1(r)),\dots,\gamma_{nn}(\sigma_n(r))) &< \sigma_n(r) \end{aligned} \tag{1.3}$$

for all $r>0$.

On $\mathbb{R}^n_+$ we employ the partial order induced by the component wise ordering, i.e., for $x,y\in\mathbb{R}^n_+$ we write $x\le y$ if $x_i\le y_i$ for all i. We write $x<y$ if $x\le y$ and $x\ne y$. To denote $x_i<y_i$ for all i we write $x\ll y$. By $x\not\ge y$ we mean the logical negation of $x\ge y$, i.e., that there exists one index i such that $x_i<y_i$.

In the linear case the above problems are of course eigenvalue problems. They can be solved if and only if the induced linear operator

$$A: x\mapsto \begin{pmatrix} \mu_1(\gamma_{11}(x_1),\dots,\gamma_{1n}(x_n)) \\ \vdots \\ \mu_n(\gamma_{n1}(x_1),\dots,\gamma_{nn}(x_n)) \end{pmatrix}$$

has spectral radius $\rho(A)<1$, where

$$\rho(A)=\max\{|\lambda| : \det(\lambda\cdot\mathrm{id}-A)=0\}$$

denotes the spectral radius of A. The solution of the path problem then essentially is an eigenvector to the dominating eigenvalue $\lambda=\rho(A)$. It is evident that the spectral radius condition is also equivalent to the global asymptotic stability of the origin in $\mathbb{R}^n_+$ with respect to the monotone system

$$x^{k+1}=Ax^k\,. \tag{1.4}$$

Lemma 1.1. *Let $A\in\mathbb{R}^{n\times n}_+$. Then the following are equivalent:*

(1) *$Ax\not\ge x$ for all $x\in\mathbb{R}^n_+, x\ne 0$;*
(2) *$\rho(A)<1$;*
(3) *there exists a vector $s\gg 0$ such that $As\ll s$;*
(4) *the origin is globally asymptotically stable with respect to* (1.4)*;*

If any of the above conditions are satisfied, then an important consequence is that $(id - A)$ *is invertible; hence* $x = Ax + b$ *is uniquely solvable for any* $b \in \mathbb{R}^n_+$*; furthermore all solutions* x *of* $(id - A)x = b$ *for* $b \geq 0$ *are nonnegative.*

Proof. Most of this is standard, we state a few references for the sake of completeness. It is well known that $A^k \to 0$ as $k \to \infty$ if and only if $\rho(A) < 1$. This gives the equivalence of (4) and (2). The theory for the equivalences (1) $\iff$ (2) $\iff$ (3) can be found in [1] as follows:

(1) $\iff$ (2) and (3) $\implies$ (1) is [1, Theorem 2.1.11, Corollary 2.1.12, p. 28]. To this end note that the implication

$$Ax \geq \alpha x \text{ for some } x > 0 \implies \rho(A) \geq \alpha$$

yields by the law of contraposition

$$\rho(A) < \alpha \implies \text{there exists no } x > 0, \text{ so that } Ax \geq \alpha x$$

where the latter can be stated equivalently as

$$Ax \not\geq \alpha x \text{ for all } x > 0.$$

And this is just (1).
(2) $\implies$ (3): If $\rho(A) < 1$ then for $\varepsilon > 0$ small enough, the matrix $\tilde{A} := A + \varepsilon E$, with $E = (e_{ij})$, $e_{ij} = 1$ for all i, j, still satisfies $\rho(\tilde{A}) < 1$ by continuity of the spectrum, but it is positive. Hence by the Perron-Frobenius Theorem [1, Theorem 2.1.3, p.27], there exists a positive eigenvector $s \gg 0$ corresponding to $\rho(\tilde{A})$, such that $As \ll \tilde{A}s = \rho(\tilde{A})s \ll s$.

$A \geq 0$ implies $A^k \geq 0$, for $k \geq 0$. So if $\rho(A) < 1$ then the inverse of $(\mathrm{id} - A)$ is given by $(\mathrm{id} - A)^{-1} = \sum_{k=0}^{\infty} A^k \geq 0$. From here it is clear that solutions x to $(\mathrm{id} - A)x = b$ are nonnegative if $b \geq 0$. □

In the more general setting these properties are still related, but not equivalent anymore, unless we add some "robustness" to our formulations. Since there is no canonical notion of spectral radius for a nonlinear monotone operator, we resort to another —equivalent to the spectral radius condition in the linear case— formulation which is also applicable for more general operators.

The problem formulations (1.2), (1.3) appear for example in the stability theory (generalized small-gain theorems) of large-scale interconnections of input-output stable (in an appropriate sense) systems, e.g., [2, 3, 15, 14, 17]. For possibly nonlinear operators on more general spaces, condition 1 in Lemma 1.1 already appears for example in Krasnosel′skiĭ's works as a sufficient condition for the existence of a fixed point, in case that the operator in question is positive [8, p. 55f],[9, p.282] and in [9, p.197] as a condition that the rotation of the vector field $x - Ax$ has rotation one on an open set containing the origin. The notion of comparison functions (class $\mathcal{K}_\infty$) has been introduced by Hahn [4, 5] and is currently widely used in the systems and control literature.

In the next section we introduce necessary notation and definitions, in Section 3 we consider the set Ω which may contain the path σ in the path construction problem, in Section 4 we consider the dynamical system induced by the monotone operator associated to the problem formulations, similar to (1.4). In Section 5 we solve the path construction problem and in Section 6 we solve the inequality problem.

2. Notation and preliminaries

2.1. Order

On the positive cone $\mathbb{R}^n_+$ we use the order notation defined in the introduction. A few of our results also hold in more general positive cones. Given a real Banach space X a *positive cone* $X_+ \subset X$ is a set satisfying $\mathbb{R}_+ \cdot X_+ \subset X_+$, $X_+ + X_+ \subset X_+$, and $X_+ \cap (-X_+) = \{0\}$. The order induced by a positive cone is given by $x \leq y$, whenever $y - x \in X_+$. As before we write $x < y$ to mean $x \leq y$ and $x \neq y$. If X_+ has nonempty interior, we write $x \ll y$, if $y - x \in \operatorname{int} X_+$. The order relation $\leq$ is closed if it is compatible with the topology on X. That is, $x \leq y$ if $y^k \to y$, $x^k \to x$ and $x^k \leq y^k$ for all k. A Banach lattice is a partially ordered Banach space $(X, \leq)$ where $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ exist for all $x, y \in X$.

Throughout the paper we assume that X_+ is a positive cone in a real Banach lattice X, compatible with the closed partial order on X.

2.2. Monotonicity

A *monotone operator* T on X_+ is a continuous mapping $T : X_+ \to X_+$ such that $x \leq y$ implies $Tx \leq Ty$. Stronger monotonicity concepts are the following. A monotone operator $T : X_+ \to X_+$ is called

1. *strictly monotone*, if $T(x) < T(y)$ for all $x, y \in X_+$ satisfying $x < y$.
2. *strongly monotone*, if $T(x) \ll T(y)$ for all $x, y \in X_+$ satisfying $x < y$;
3. *strictly increasing*, if $T(x) \ll T(y)$ for all $x, y \in X_+$ satisfying $x \ll y$;
4. *eventually strongly monotone*, if there exists a $k_0 > 0$ such that $T^k(x) \ll T^k(y)$ for all $k \geq k_0$ and all $x, y \in X_+$ satisfying $x < y$;
5. *strongly order-preserving* (SOP), if for $x < y$ there exist respective neighborhoods U, V of x, y and $k_0 \geq 1$ such that $T^k(U) \leq T^k(V)$ for all $k \geq k_0$.

We have the following obvious implications: 2. $\implies$ 3., 2. $\implies$ 1., and 2. $\implies$ 4.

2.3. No-joint-increase condition

The *no-joint-increase condition* is given by

$$T(x) \not\geq x, \quad \text{for all } x \neq 0. \tag{2.1}$$

It is not difficult to see that a monotone operator $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ must satisfy $T(0) = 0$ if (2.1) holds.

Lemma 2.1. *Let $T : X_+ \to X_+$ be a monotone operator satisfying $T(x) \not\geq x$ for all $x \neq 0$. Then $T^k(x) \not\geq x$ for all $x \neq 0$ and $k \geq 1$.*

Proof. Assume the contrary: Suppose there exist an $x \neq 0$ and a $k > 1$, such that $T^k(x) \geq x$. Let $z = \sup\{x, T(x), \dots, T^{k-1}(x)\}$. Clearly $z \neq 0$. Hence

$$T(z) \geq \sup\{T(x), \dots, T^k(x)\}$$

by monotonicity and

$$= \sup\{x, T(x), \dots, T^k(x)\}$$

since $T^k(x) \geq x$, which also implies

$$\geq \sup\{x, T(x), \dots, T^{k-1}(x)\} = z$$

contradicting $T(z) \not\geq z$ for all $z \neq 0$. Hence no such x and k exist, i.e., for all $k \geq 1$ and $x \neq 0$ we have $T^k(x) \not\geq x$. □

2.4. $\mathcal{K}_\infty$ Paths

For the path parametrization problem we fix the following notation. A $\mathcal{K}_\infty^n$-path σ, written also as $\sigma \in \mathcal{K}_\infty^n$, denotes the *parametrized path*

$$r \mapsto \begin{pmatrix} \sigma_1(r) \\ \vdots \\ \sigma_n(r) \end{pmatrix} \in \mathbb{R}_+^n, \quad r \in \mathbb{R}_+, \tag{2.2}$$

where each $\sigma_i \in \mathcal{K}_\infty$.

2.5. Projections

Let I be an index set such that $\emptyset \neq I \subset \{1, \dots, n\}$ and denote by $\#I$ the cardinality of I. Let $(e_k)_{k=1,\dots,n}$ be the standard basis of $\mathbb{R}^n$. On $\mathbb{R}_+^n$ we denote by $\pi_I : \mathbb{R}_+^n \to \mathbb{R}_+^{\#I}$ the *projection* of the coordinates in $\mathbb{R}_+^n$ corresponding to the indices in I onto $\mathbb{R}^{\#I}$. The corresponding *anti-projection* is $\iota_I : \mathbb{R}_+^{\#I} \to \mathbb{R}_+^n$ mapping $x \mapsto (x_1 e_{i_1} + \ldots + x_{\#I} e_{i_{\#I}})$, where we write $I = \{i_1, \dots, i_{\#I}\}$. For short we write π_i and ι_i instead of $\pi_{\{i\}}$ and $\iota_{\{i\}}$ for $i = 1, \dots, n$. For any index set $\emptyset \neq I \subset \{1, \dots, n\}$ and vector $x \in \mathbb{R}_+^n$ we denote by $x|_I$ the vector in $\mathbb{R}_+^n$ with elements

$$(x|_I)_i = \begin{cases} x_i & \text{if } i \in I \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

2.6. Induced operators

To express the two problems under consideration in a vector notation, we introduce the following operator notation.

Definition 2.2 (Monotone aggregation functions). *A function $\mu : \mathbb{R}_+^n \to \mathbb{R}_+$ is called a* monotone aggregation function (MAF_n) *if μ is continuous and satisfies*

(M1) *positivity: $\mu(s) \geq 0$ for all $s \in \mathbb{R}_+^n$;*
(M2) *strictly increasing*[1]*: if $x \ll y$ then $\mu(x) < \mu(y)$.*

[1] Cf. Assumption 2.3, where for the purposes of this paper (M2) is further restricted.

By $\mu \in \mathrm{MAF}_n^m$ *we denote vector monotone aggregation functions, i.e.,* $\mu_i \in \mathrm{MAF}_n$ *for* $i = 1, \dots, m$*. If* $m = n$ *we simply write* MAF^n *instead of* MAF_n^n*.*

A direct consequence of (M2) and continuity is that also the weaker monotonicity property $x \le y \implies \mu(x) \le \mu(y)$ holds for MAFs. For some results we will additionally require the following properties:

(M3) *unboundedness: if* $\|x\| \to \infty$ *then* $\mu(x) \to \infty$*;*
(M4) *sub-additivity:* $\mu(x+y) \le \mu(x) + \mu(y)$*.*

Examples of monotone aggregation functions (which also satisfy M3,M4) include monotone norms on $\mathbb{R}_+^n$ (that in turn includes all p-norms). In particular

$$\Sigma:\ (x_1,\dots,x_n)^T \mapsto \sum_{i=1}^n x_i \tag{2.4}$$

and

$$\oplus:\ (x_1,\dots,x_n)^T \mapsto \max_{i=1,\dots,n} x_i \tag{2.5}$$

are examples of monotone aggregation functions. The vector MAF's induced by (2.4),(2.5) will be again denoted by Σ and $\oplus$.

Given class $\mathcal{G}$ functions γ_{ij}, $i,j = 1,\dots,n$, we can arrange these functions in a matrix $\Gamma = (\gamma_{ij})$. Together with $\mu \in \mathrm{MAF}^n$ this matrix gives rise to a monotone operator $\Gamma_\mu : \mathbb{R}_+^n \to \mathbb{R}_+^n$,

$$\Gamma_\mu(x) := \begin{pmatrix} \mu_1(\gamma_{11}(x_1),\dots,\gamma_{1n}(x_n)) \\ \vdots \\ \mu_n(\gamma_{n1}(x_1),\dots,\gamma_{nn}(x_n)) \end{pmatrix}, \quad x \in \mathbb{R}_+^n\,. \tag{2.6}$$

This Γ_μ operator can be thought of as the composition of two operators,

$$\mathbb{R}_+^n \xrightarrow{\ \Gamma\ } \mathbb{R}_+^{n\times n} \xrightarrow{\ \mu\ } \mathbb{R}_+^n\,.$$

Throughout the paper we will make the following compatibility assumption between matrices Γ and MAFs μ:

Assumption 2.3 (Standard assumption). *Given* $\Gamma \in \mathcal{G}^{n\times n}$ *and* $\mu \in \mathrm{MAF}^n$*, we will from now on assume that* Γ *and* μ *are* compatible *in the following sense: For each* $i = 1,\dots,n$*, let* I_i *denote the set of indices corresponding to the nonzero entries in the* i*th row of* Γ*. Then it is understood that also the restriction of* μ_i *to the indices* I_i *satisfies (M2), i.e., if* $x|_{I_i} \ll y|_{I_i}$ *then* $\mu_i(x|_{I_i}) < \mu_i(y|_{I_i})$*.*

2.7. Factorizing fudge factors

Frequently we will use diagonal operators. We write $D = \mathrm{diag}(\alpha)$ for some class $\mathcal{K}_\infty$ function α, to denote the mapping

$$x \mapsto \begin{pmatrix} \alpha(x_1) \\ \vdots \\ \alpha(x_n) \end{pmatrix}, \quad x \in \mathbb{R}_+^n\,, \tag{2.7}$$

i.e., the function α is applied to every component of x. Three technical lemmas will be needed later on, cf. [16].

Lemma 2.4. *Let $\rho \in \mathcal{K}_\infty$. Then there exists a $\tilde{\rho} \in \mathcal{K}_\infty$ such that $(id+\rho)^{-1} = id-\tilde{\rho}$.*

Proof. Just define $\tilde{\rho} = \rho\circ(\mathrm{id}+\rho)^{-1}$. Then $(\mathrm{id}-\tilde{\rho})\circ(\mathrm{id}+\rho) = (\mathrm{id}+\rho)-\tilde{\rho}\circ(\mathrm{id}+\rho) = \mathrm{id} + \rho - \rho \circ (\mathrm{id} + \rho)^{-1} \circ (id + \rho) = \mathrm{id} + \rho - \rho = \mathrm{id}$, which proves the lemma. □

Lemma 2.5.
1. *Let $D = \mathrm{diag}(\rho)$ for some $\rho \in \mathcal{K}_\infty$ such that $\rho > id$. Then for any $k \geq 0$ there exist $\rho_1^{(k)}, \rho_2^{(k)} \in \mathcal{K}_\infty$ satisfying $\rho_i^{(k)} > id$, such that for $D_i^{(k)} = \mathrm{diag}(\rho_i^{(k)})$, $i = 1, 2$,*
$$D = D_1^{(k)} \circ D_2^{(k)}.$$
Moreover, $D_2^{(k)}$, $k \geq 0$, can be chosen such that for all $0 \ll s \in \mathbb{R}^n_+$ we have
$$D_2^{(k)}(s) \ll D_2^{(k+1)}(s).$$
2. *Let $D = \mathrm{diag}(id + \alpha)$ for some $\alpha \in \mathcal{K}_\infty$. Then there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, such that for $D_i = \mathrm{diag}(id + \alpha_i)$, $i = 1, 2$,*
$$D = D_1 \circ D_2.$$

Proof. Both assertions are componentwise applications of the following considerations:

Consider the first assertion. For $\lambda^{(k)} \in (0,1)$ let $\rho_1^{(k)} := \lambda^{(k)}\rho + (1 - \lambda^{(k)})\mathrm{id}$. Clearly $\rho_1^{(k)} \in \mathcal{K}_\infty$ and $\mathrm{id} < \rho_1^{(k)} < \rho$. This implies that $\rho_2^{(k)} := (\rho_1^{(k)})^{-1} \circ \rho > \mathrm{id}$. As the composition of $\mathcal{K}_\infty$ functions, $\rho_2^{(k)}$ is also of class $\mathcal{K}_\infty$. Clearly $\lambda^{(k+1)} < \lambda^{(k)}$ implies $\rho_1^{(k+1)}(t) < \rho_1^{(k)}(t)$ and $\rho_2^{(k+1)}(t) > \rho_2^{(k)}(t)$ for $t > 0$. Now choose, e.g., $\lambda^{(k)} := \frac{1}{k+2}$ to obtain a strictly decreasing sequence $\{\lambda^{(k)}\}_{k\geq 0}$ satisfying $\lambda^{(k)} \in (0, \frac{1}{2}]$ for all $k \geq 0$. This implies the first assertion.

For the second assertion of the lemma note that $(\frac{1}{2}\alpha + id)$ is of class $\mathcal{K}_\infty$. Now choose, e.g., $\alpha_2 = \frac{1}{2}\alpha$ and $\alpha_1 = \frac{1}{2}\alpha \circ (\frac{1}{2}\alpha + id)^{-1}$. Then
$$(\alpha_1 + \mathrm{id}) \circ (\alpha_2 + \mathrm{id}) = \alpha_1(\alpha_2 + \mathrm{id}) + \alpha_2 + \mathrm{id} =$$
$$\frac{1}{2}\alpha \circ (\frac{1}{2}\alpha + id)^{-1} \circ (\frac{1}{2}\alpha + \mathrm{id}) + \frac{1}{2}\alpha + id = \frac{1}{2}\alpha \circ \mathrm{id} + \frac{1}{2}\alpha + \mathrm{id} = \alpha + \mathrm{id}\,.$$
This yields the second assertion. □

We note an easy consequence.

Lemma 2.6. *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$. Let $D = \mathrm{diag}(\rho)$ for some $\rho \in \mathcal{K}_\infty$. Then the following are equivalent:*
1. *For all $x \neq 0$*
$$D \circ T(x) \not\geq x;$$
2. *For all $x \neq 0$*
$$T \circ D(x) \not\geq x;$$

3. *For all* $x \neq 0$

$$D_2 \circ T \circ D_1(x) \not\geq x,$$

where D *is factorized into* $D = D_1 \circ D_2$ *by Lemma 2.5.*

Proof. Note that D is invertible and, as T, monotone. Moreover, also D^{-1} is monotone. Assume for all $x \neq 0$ we have $D \circ T(x) \not\geq x$. Every $x \geq 0$ can be written as $x = D(y)$ for some $y \in \mathbb{R}^n_+$. Then $D \circ T(D(y)) \not\geq D(y)$, and application of D^{-1}, which preserves inequalities, yields $T(D(y)) \not\geq y$, for all $y \neq 0$. This proves equivalence of the first two properties, the third is derived analogously using additionally Lemma 2.5. □

2.8. Graphs

A (finite) directed graph $G = (V, E)$ consists of a finite set of vertices V and a set of edges $E \subset V \times V$. In case of a graph with n vertices we may identify $V = \{1, \ldots, n\}$. The adjacency matrix $A_G = (a_{ij})$ of this graph is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } (j,i) \in E, \\ 0 & \text{else.} \end{cases}$$

Conversely, given an $n \times n$-matrix A, a graph $G(A) = (V, E)$ is defined by $V := \{1, \ldots, n\}$ and $E = \{(j,i) \in V \times V : a_{ij} \neq 0\}$.

We say A_G is *irreducible*, if G is *strongly connected*, that is, for every pair of vertices (i, j) there exists a sequence of edges (a *path*), $\big((i_0, i_1), (i_1, i_2), \ldots, (i_{k-1}, i_k)\big)$, with $i_0 = i$, $i_k = j$ for some $k \geq 1$, and $(i_{l-1}, i_l) \in E$ for $l = 1, \ldots, k$, connecting vertex i to vertex j. Then k is the *length* of the path. Obviously A_G is irreducible if and only if its transpose A_G^T is. The adjacency matrix A_G is *reducible* if it is not irreducible. In this case A_G can be written as

$$P^T \cdot \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \cdot P \tag{2.8}$$

where P is a permutation matrix, and A_{11} and A_{22} are square matrices [1]. In terms of the graph G the similarity transformation (i.e., multiplication of A_G with a permutation matrix) is a renumbering of the vertices.

The matrix A_G is *primitive*, if there exists a positive integer m such that $(A_G)^m$ has only positive entries. In terms of the graph G this means that there exists a number m, such that for any pair of vertices there exists a path of length m connecting these vertices.

Let M be an $n \times n$ matrix, whose elements can be distinguished to be either positive or zero. For example, M could be a $\mathcal{G}^{n \times n}$-matrix. In this case positive entries correspond to strictly increasing functions, zero entries correspond to functions that are constantly zero, and negative entries do not occur. Define the adjacency matrix A of M by

$$a_{ij} = \begin{cases} 1 & \text{if } m_{ij} \text{ is positive,} \\ 0 & \text{else.} \end{cases} \tag{2.9}$$

The matrix M is called irreducible/ reducible/ primitive if and only if A is irreducible/ reducible/ primitive.

2.9. Transformations

Let $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$. If Γ is reducible, then a permutation $P \in \{0,1\}^{n\times n}$, $P = (p_{ij})_{i,j=1}^n$, transforms it into a block upper triangular matrix $P\Gamma P^T$, defined by $(P\Gamma P^T)_{ij} = \sum_{k,l} p_{ki} p_{lj} \gamma_{kl}$, where the blocks on the diagonal are either irreducible or zero 1×1-blocks. This transformation of Γ by a permutation matrix only rearranges the positions of the class $\mathcal{K}_\infty$-functions, and the result is again a matrix in $\mathcal{G}^{n\times n}$. The product $P\Gamma P^T = \tilde{\Gamma}$ is called *similarity transformation* of Γ.

In terms of operators a similarity transform is a change of coordinates, more precisely, a renumbering of the standard basis vectors. If we want to consider Γ_μ in the new coordinates, we also have to transform μ to $\tilde{\mu} := P \circ \mu \circ P^T$. Here we identify the symbols P and P^T with their induced linear operators, i.e., P^T maps vectors $s \in \mathbb{R}_+^n$ to $P^T s$. This yields an operator $\tilde{\Gamma}_{\tilde{\mu}}$, which is Γ_μ in the changed coordinates:

$$\tilde{\Gamma}_{\tilde{\mu}} = (P \circ \mu \circ P^T) \circ (P \circ \Gamma \circ P^T) = P \circ \Gamma_\mu \circ P^T \tag{2.10}$$

A straight forward result relating graph or, respectively, matrix structure to monotonicity is the next one.

Lemma 2.7. *Let $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$. Then we have*

1. *Γ_μ is strictly increasing if and only if Γ has no zero rows; this is the case, e.g., for irreducible Γ;*
2. *Γ_μ is eventually strongly monotone if Γ is primitive and μ is strongly monotone (i.e., for all i: $\mu_i(u) < \mu_i(v)$ whenever $u < v$).*

Proof. 1. Let $u, v \in \mathbb{R}_+^n$, $v \gg 0$. Then $u \ll u + v$ and if Γ has no zero rows

$$\Gamma_\mu(u+v) = \begin{pmatrix} \mu_1(\gamma_{11}(u_1+v_1), \dots, \gamma_{1n}(u_n+v_n)) \\ \vdots \\ \mu_n(\gamma_{n1}(u_1+v_1), \dots, \gamma_{nn}(u_n+v_n)) \end{pmatrix} \gg \begin{pmatrix} \mu_1(\gamma_{11}(u_1), \dots, \gamma_{1n}(u_n)) \\ \vdots \\ \mu_n(\gamma_{n1}(u_1), \dots, \gamma_{nn}(u_n)) \end{pmatrix} = \Gamma_\mu(u),$$

since for each row $i \in \{1, \dots, n\}$ for at least one j the (i,j)th entry of Γ is of class $\mathcal{K}_\infty$, i.e., strictly increasing, so $\gamma_{ij}(u_j + v_j) > \gamma_{ij}(u_j)$. Now each μ_i is strictly increasing with respect to the set of nonzero entries in the ith row of Γ, hence $\mu_i(\dots, \gamma_{ij}(u_j+v_j), \dots) > \mu_i(\dots, \gamma_{ij}(u_j), \dots)$. Therefore Γ_μ is strictly increasing. If the ith row of Γ is a zero row, then there is no such (i,j)th entry of Γ of class $\mathcal{K}_\infty$. This proves the first part.

2. Let $A = (a_{ij})$ be the adjacency matrix corresponding to the graph defined by Γ, i.e., $a_{ij} = 1$ iff $\gamma_{ij} \in \mathcal{K}_\infty$. By assumption there exists a $k > 0$ such that A^k has only positive entries.

We now show that the (i,j)th entry of A^k is zero if and only if $\pi_i \circ (\Gamma_\mu)^k \circ \iota_j \equiv 0$. This implies the claim.

To this end note that $\pi_i \circ (\Gamma_\mu)^k \circ \iota_j$ is strictly increasing if there exist $i = l_1, \ldots, l_{k+1} = j \in \{1, \ldots, n\}$, such that $\pi_{l_m} \circ \Gamma_\mu \circ \iota_{l_{m+1}}$ is strictly increasing, for $m = 1, \ldots, k$. This follows easily by induction on k. Now $\pi_{l_m} \circ \Gamma_\mu \circ \iota_{l_{m+1}}$ is strictly increasing if and only if $a_{l_m l_{m+1}} = 1$. Then

$$(A^k)_{ij} = \sum_{\nu_1, \ldots, \nu_{k-1}=1}^{n} \underbrace{a_{i\nu_1} a_{\nu_1 \nu_2} \cdots a_{\nu_{k-1} j}}_{\geq 0} \neq 0$$

implies that there exist $i = l_1, \ldots, l_{k+1} = j \in \{1, \ldots, n\}$, such that $a_{l_m l_{m+1}} = 1$ for $m = 1, \ldots, k$. This proves the claim. □

3. Decay sets

Given a monotone operator $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ we define the *sets of decay*

$$\Omega(T) := \{x \in \mathbb{R}^n_+ : T(x) \ll x\}; \tag{3.1}$$

$$\Psi(T) := \{x \in \mathbb{R}^n_+ : T(x) \leq x\}. \tag{3.2}$$

For short we just write Ω or Ψ if the reference to T is clear from the context. In general we have $\Psi \neq \overline{\Omega}$, e.g., for the identity map. But we always have $\overline{\Omega} \subset \Psi$.

A set $A \subset \mathbb{R}^n_+$ will be called *radially unbounded*, if for any $x \in \mathbb{R}^n_+$ there is a $y \in A$ so that $x \leq y$. As usual, a set $A \subset \mathbb{R}^n_+$ is called *unbounded* if for any $r \geq 0$ there exists $y \in A$ with $\|y\| > r$. Clearly, radial unboundedness implies unboundedness.

Example 3.1. *For $T = \frac{1}{2} id$ on $\mathbb{R}^n_+$ we have $\Psi = \mathbb{R}^n_+$ and $\Omega = \operatorname{int} \mathbb{R}^n_+$. Both sets are radially unbounded.*

In order to investigate the relation between decay sets and the no-joint-increase condition (2.1), we define the sets

$$\Omega_i := \Omega_i(T) = \{x \in \mathbb{R}^n_+ : T(x)_i < x_i\}. \tag{3.3}$$

First we state an obvious result.

Lemma 3.2 ([2]). *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a monotone operator. Then $T(x) \not\geq x$ for all $x \neq 0$ if and only if*

$$\left(\bigcup_{i=1}^n \Omega_i\right) \cup \{0\} = \mathbb{R}^n_+.$$

Lemma 3.2 regards the union of the sets Ω_i. A much more involved result concerns the intersection of the sets Ω_i. Take

$$S_r := \left\{ s \in \mathbb{R}^n_+ : \sum_i s_i = r \right\} = \left\{ x \in \mathbb{R}^n_+ : \|x\|_1 = r \right\} \tag{3.4}$$

which is an $(n-1)$-simplex in $\mathbb{R}^n_+$ of distance r to the origin. This simplex will play an important role in the sequel. The next theorem is slightly more general than the corresponding version in [2].

Theorem 3.3. *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a monotone operator. Let $r > 0$ and assume that $T(x) \not\geq x$ for all $x \in S_r$. Then*

$$\left(\bigcap_{i=1}^n \Omega_i\right) \cap S_r \neq \emptyset,$$

and $x \in \bigcap_{i=1}^n \Omega_i \cap S_r$ implies $x \gg 0$. In particular, if $T(x) \not\geq x$ for all $x \neq 0$ then the set Ω is unbounded.

The proof is very similar to the version in [2] and requires the use of a topological fixed point theorem due to Knaster, Kuratowski, and Mazurkiewicz from 1929 [6, 11, 7]. Combining Lemma 2.1 and Theorem 3.3 we obtain:

Corollary 3.4. *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a monotone operator. Then $T(x) \not\geq x$ for all $x \neq 0$ implies that for any $r > 0$, $k \geq 1$,*

$$\Omega(T^k) \cap S_r \neq \emptyset.$$

Now we give two catalogs of properties of the sets Ψ and Ω. These catalogs will help to better understand geometric properties of Ψ and Ω, especially when T has more structure than just monotonicity.

Lemma 3.5. *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a monotone operator. Then we have*

1. *$T^{k+1}(\Psi) \subset T^k(\Psi) \subset \Psi$ for $k \geq 0$;*
2. *$T^k(\Omega) \subset \Omega$ for $k \geq 0$ if T is strictly increasing;*
3. *$T^k(\Psi \setminus \{0\}) \subset \Omega$ for $k \geq 1$ if T is strongly monotone and $T(x) \neq x$ for all $x \neq 0$;*
4. *there exists a $k_0 \geq 1$ such that $T^k(\Psi \setminus \{0\}) \subset \Omega$ for $k \geq k_0$ if T is eventually strongly monotone.*

Proof.

1. Let $x \in \Psi$, i.e., $T(x) \leq x$. By monotonicity of T we find $T(T(x)) \leq T(x) \leq x$ and inductively $T^{k+2}(x) \leq T^{k+1}(x) \leq T^k(x) \leq x$ for all $k \geq 0$, i.e. $T^{k+1}(x) \in T^k(\Psi)$ and $T^k(x) \in \Psi$.
2. Let $x \in \Omega$, then $T(x) \ll x$. Applying the strictly increasing operator T gives $T^2(x) \ll T(x) \ll x$ and inductively $T(T^k(x)) \ll T^k(x)$, i.e., $T^k(x) \in \Omega$.
3. Let $x \in \Psi$, $x \neq 0$. Then $0 < x$ and $0 \leq T(0) \ll T(x) < x$ as $T(x) \neq x$. For $k \geq 1$ an application of the strongly monotone operator T^k to $0 \ll T(x) < x$ gives $T(T^k(x)) \ll T^k(x)$, that is, we have $T^k(x) \in \Omega$.
4. Since T is eventually strongly monotone, there exists a $k_0 \geq 1$ such that $x < y$ implies $T^k(x) \ll T^k(y)$. The rest of this proof goes along the lines of part 3 for $k \geq k_0$.

□

The second catalog involves a diagonal operator D as defined in Subsection 2.7. Such diagonal operators will play a crucial role for the path construction problem.

Lemma 3.6. *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a monotone operator. Let $\rho \in \mathcal{K}_\infty$, $\rho > id$, and $D = \operatorname{diag}(\rho)$. Then we have*

1. *$\Psi(D \circ T) \setminus \{0\} \subset \Omega(T)$, if T is strongly monotone;*
2. *$\Psi(D \circ T) \cap \{s \in \mathbb{R}^n_+ : s \gg 0\} \subset \Omega(T)$, if T is strictly increasing;*
3. *$T\left(\Psi(D \circ T) \setminus \{0\}\right) \subset \Omega(T)$, provided that T is strongly monotone.*

All results are also true with $D \circ T$ replaced by $T \circ D$.

Proof. 1. Let $x \in \Psi(D \circ T) \setminus \{0\}$. Then $0 < x$ implies $0 \ll T(x) \ll D(T(x)) \leq x$ and hence $x \in \Omega(T)$. Now let $x \in \Psi(T \circ D) \setminus \{0\}$. Since $0 < x < D(x)$ we have $0 \ll T(x) \ll T(D(x)) \leq x$, so $x \in \Omega(T)$.

2. Let $x \in \Psi(T \circ D) \cap \{s \in \mathbb{R}^n_+ : s \gg 0\}$. Again $0 \ll x \ll D(x)$ implies $0 \ll T(x) \ll T(D(x)) \leq x$, hence $x \in \Omega(T)$. Let $x \in \Psi(D \circ T) \cap \{s \in \mathbb{R}^n_+ : s \gg 0\}$. Then $0 \ll x$ gives $0 \ll T(x) \ll D(T(x)) \leq x$, hence $x \in \Omega(T)$.
3. This result is implied by 1. and Lemma 3.5.2. □

Observe that in general the pre-image $T^{-1}[\Psi]$ is not a subset of Ψ:

Example 3.7. *Let $A = \begin{pmatrix} 0.2 & 0 \\ 0.8 & 0.3 \end{pmatrix}$, so we have $\rho(A) = 0.3$. For $x = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ we find $Ax = \begin{pmatrix} 0.1 \\ 0.55 \end{pmatrix} \not\leq x$, but $A^2 x = \begin{pmatrix} 0.02 \\ 0.245 \end{pmatrix} \ll Ax$. Hence $Ax \in \Psi$ but $x \notin \Psi$.*

4. Induced dynamical system

Given a monotone operator $T : X_+ \to X_+$, its induced dynamical system is

$$x^{k+1} = T(x^k)\,. \tag{4.1}$$

The origin in X_+ is *attractive* with respect to (4.1), if there exists a neighborhood $U \subset X_+$ of the origin, so that for all $x^0 \in U$, $x^k \to 0$ as $k \to 0$. The *region of attraction* of the origin is $R_0 = \{x^0 \in X_+ : x^k \to 0 \text{ as } k \to \infty\}$. If $R_0 = X_+$ then the origin is said to be *globally attractive*. The origin is *stable* with respect to (4.1), if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $x^0 \in X_+ : \|x^0\| < \delta$ implies $\|x^k\| < \varepsilon$ for all $k \geq 0$. The origin is *globally asymptotically stable* (GAS), if it is globally attractive and stable. The *orbit* of $x \in X_+$ is $O(x) := \{T^k(x)\}_{k \geq 0}$ and the *omega limit set* of x is $\omega(x) := \bigcap_{k \geq 0} \overline{O(T^k(x))}$.

Proposition 4.1. *Let $T : X_+ \to X_+$ be a monotone operator. If the origin is attractive with respect to (4.1), then for all $x \in R_0 \setminus \{0\}$ it holds that $T(x) \not\geq x$.*

Proof. We argue indirectly. Let $x \in R_0$, $x \neq 0$, and assume $T(x) \geq x$. Then by monotonicity $T^{k+1}(x) \geq T^k(x) \geq x$ for all $k \geq 0$. Hence $T^k(x) \not\to 0$ as $k \to \infty$, contradicting our assumption that $x \in R_0$. Therefore we must have $T(x) \not\geq x$ for all $x \in R_0$, $x \neq 0$. □

As a consequence of this result and Theorem 3.3 it follows that the origin is stable with respect to a monotone system, if it is attractive.

Proposition 4.2. *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a monotone operator. Suppose that the origin is attractive with respect to* (4.1)*. Then the origin is also stable.*

Proof. By the preceding Proposition $T(x) \not\geq x$ for all $x \in R_0$, $x \neq 0$. Hence for a given $\varepsilon > 0$ we may pick an $r \in (0, \varepsilon]$ small enough such that $S_r \subset R_0$ and by Theorem 3.3 choose a point $y \in \Omega(T) \cap S_r$. This point satisfies $y \gg 0$. Let

$$\delta := \sup\{d \in \mathbb{R}_+ : x \ll y \ \forall x : \ \|x\| < d\}.$$

Note that $\delta > 0$ and that $\|x\| < \delta$ implies $x \ll y$. By monotonicity this implies $T^k(x) \leq T^k(y) \ll y$ for $k \geq 1$, hence $\|T^k(x)\|_1 < r \leq \varepsilon$, proving stability of the origin. □

Putting the pieces together, we obtain the following result.

Theorem 4.3. *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a monotone operator. Then the origin is locally asymptotically stable for the discrete time system* (4.3) *if and only if T satisfies $T(x) \not\geq x$ for all $x \in U$, $x \neq 0$ in a neighborhood U of the origin.*

Note that the global no-joint-increase condition (2.1) for a monotone operator T on $\mathbb{R}^n_+$ is not sufficient to ensure global asymptotic stability of the origin with respect to the monotone system induced by T.

Example 4.4. *Fix real constants $\lambda \in (0,1)$ and $\mu \geq 0$. Let $T : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ be given by*

$$T\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \lambda s_1 + s_1^2 s_2 + \mu s_2 \\ \lambda s_2 \end{pmatrix}$$

for all $s = (s_1, s_2)^T \in \mathbb{R}^2_+$. The map T satisfies $T(0) = 0$ and is continuous and monotone. It is easy to see that it satisfies $T(s) \not\geq s$ for all $s \neq 0$. If $s^0 = (s_1^0, s_2^0)^T \gg 0$ is such that $s_1^0 > 1/(\lambda s_2^0)$ then the trajectory of (4.3) *starting in s^0 is unbounded in the first component: The condition $s_1(k) > 1/(\lambda s_2(k))$ with $s(k) > 0$ implies $s_1(k+1) = s_1(k)(\lambda + s_1(k)s_2(k) + \mu s_2(k)) > s_1(k)/\lambda > 1/(\lambda^2 s_2(k)) = 1/(\lambda s_2(k+1))$ and clearly $s_2(k+1) > 0$. By induction we obtain a trajectory that converges to* 0 *in the second component and diverges in the first one. Hence for the monotone system induced by T the origin is not globally asymptotically stable.*

The example above shows that the no-joint-increase condition alone is not enough to deduce global asymptotic stability of the origin with respect to the monotone system induced by T. But a related result is in fact true, at least if we restrict our attention to operators of the form $T = \Gamma_\mu$ and additionally impose a restriction on the class of matrices Γ.

Theorem 4.5. *Let $\mu \in \mathrm{MAF}^n_n$ satisfy (M3) and let $\Gamma \in \mathcal{G}^{n\times n}$. Assume $T := \Gamma_\mu : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ satisfies* (2.1) *assume further that Γ is irreducible. Then the origin is globally asymptotically stable with respect to* (4.1)*.*

This result will easily follow from Theorem 5.5 in Section 5. Instead of restricting the class of matrices Γ, we may as well introduce "fudge factors" to the no-joint-increase condition and the dynamical system under consideration.

Theorem 4.6. *Let $\Gamma = (\gamma_{ij}) \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$ satisfying (M3),(M4). Then the following are equivalent:*

1. *There exists a $D = \mathrm{diag}(id + \rho)$, $\rho \in \mathcal{K}_\infty$, such that*
$$D \circ \Gamma_\mu(x) \not\geq x \text{ for all } x \neq 0\,. \tag{4.2}$$
2. *There exists a $\tilde{D} = \mathrm{diag}(id + \tilde{\rho})$, $\tilde{\rho} \in \mathcal{K}_\infty$, such that the origin is globally asymptotically stable with respect to*
$$x^{k+1} = \tilde{D} \circ \Gamma_\mu(x^k)\,. \tag{4.3}$$

The proof is kept short by using results from Section 5.

Proof. Assume that the origin is globally asymptotically stable with respect to (4.3), i.e., that the second statement holds true. By Proposition 4.1 for $D = \tilde{D}$ the first statement follows.

Now assume the first statement of the theorem is true. Using Lemma 2.5 write $D = D_1 \circ D_2$ so that we have $D_2 \circ \Gamma_\mu \circ D_1(x) \not\geq x$ for all $x \neq 0$ by Lemma 2.6. Observe that the operator $\tilde{\Gamma} := \Gamma \circ D_1 : \mathbb{R}_+^n \to \mathbb{R}_+^{n\times n}$ corresponds to the $\mathcal{G}^{n\times n}$ matrix $\tilde{\Gamma} = (\gamma_{ij} \circ (\mathrm{id} + \rho_1))_{ij}$. Apply Theorem 5.10 to D_2 and $\tilde{\Gamma}_\mu$ to obtain $\sigma \in \mathcal{K}_\infty^n$ such that for all $r > 0$,

$$(\Gamma_\mu \circ D_1)(\sigma(r)) = \tilde{\Gamma}_\mu(\sigma(r)) \ll \sigma(r),$$

or, in other words, $\sigma(r) \in \Omega(\Gamma_\mu \circ D_1)$. It follows that Ψ is radially unbounded and by Lemma 5.1 the origin is globally asymptotically stable with respect to

$$x^{k+1} = (\Gamma_\mu \circ D_1)(x^k)\,. \tag{4.4}$$

Hence, the origin is also stable with respect to (4.3), where $\tilde{D} = D_1$. □

5. Ω-paths

In this section we provide a solution for the path parametrization problem (1.3). In the language of Section 3, we aim to construct a $\mathcal{K}_\infty^n$-path σ with image in the decay set $\Omega(\Gamma_\mu) \cup \{0\}$ (hence the name Ω-path), where the monotone operator Γ_μ is given by the problem formulation (1.3). A prerequisite for the path construction is of course that the set $\Omega(\Gamma_\mu)$ is radially unbounded, for otherwise there is no hope to construct a path parametrization with unbounded component functions. Obviously, we also have to have $\Omega(T) \cap S_r \neq \emptyset$, for all $r > 0$. From here it follows easily with the next lemma, that Γ_μ must satisfy the no-joint-increase condition (2.1).

Lemma 5.1. *Let $T : \mathbb{R}_+^n \to \mathbb{R}_+^n$ be a monotone operator. Let $\Psi(T)$ be radially unbounded and $T(0) = 0$ be the only fixed point of T. Then the origin is globally asymptotically stable with respect to* (4.1). *In particular, T also satisfies* (2.1).

Proof. Just observe that by radial unboundedness of Ψ, for any $x^0 \in \mathbb{R}^n_+$, there exists a $y^0 \in \Psi$, such that $x^0 \le y^0$. By monotonicity of T the trajectory starting in x^0 is dominated by the trajectory starting at y^0, i.e., $x^k \le y^k$ for all $k \ge 0$. The decay set Ψ is forward invariant, so $y^k \in \Psi(T)$ for all $k \ge 0$. Since the sequence $\{y^k\}$ is bounded, it must have an accumulation point. By continuity of T this accumulation point must be a fixed point. This proves attractivity. The claim now follows from Propositions 4.1 and 4.2. □

From Example 4.4 it follows that for general monotone operators $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ the decay set $\Omega(T)$ need not to be radially unbounded, even if T satisfies the no-joint-increase condition and hence Ω is unbounded. So stronger assumptions have to be imposed in order to be able to prove the existence of a $\mathcal{K}^n_\infty$-path σ satisfying (1.3).

Using a continuous selection theorem, the existence of *a* path $\nu : \mathbb{R}_+ \to \mathbb{R}^n_+$ with image in the set $\Omega(T) \cup \{0\}$ for such arbitrary monotone operators $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ satisfying (2.1) is at hand. However, in this way in general the component functions ν_i are not easily guaranteed to be strictly increasing or unbounded.

Proposition 5.2. *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a strictly increasing operator satisfying* (2.1). *Then for any $s \in \Omega(T)$ there exists a continuous path $\sigma : [0,1] \to \mathbb{R}^n_+$, so that $T(\sigma(r)) \ll \sigma(r)$ for all $r \in (0,1]$, each σ_i is strictly increasing, and $\sigma(0) = 0$, $\sigma(1) = s$. Moreover, σ can be chosen to be piecewise linear on* $(0,1]$.

Proof. Observe that $T^k(s) \gg T^{k+1}(s) \to 0$, and in particular $T^k(s) \gg 0$ for all $k \ge 0$. For $\lambda \in (0,1)$ the convex combination $(1-\lambda)T^{k+1}(s) + \lambda T^k(s)$ satisfies

$$T((1-\lambda)T^{k+1}(s) + \lambda T^k(s)) \ll T^{k+1}(s) \ll (1-\lambda)T^{k+1}(s) + \lambda T^k(s)\,,$$

so that we obtain a componentwise strictly increasing path from $T^{k+1}(s)$ to $T^k(s)$ by linear interpolation. As a consequence, each σ_i can be chosen to be piecewise linear on $(0,1]$. Now define $\sigma(0) = 0$ to complete the proof. □

If T is bounded, then it is easy to extend the path σ in the previous result to a $\mathcal{K}_\infty$-path $\tilde{\sigma}$ satisfying $T(\tilde{\sigma}(r)) \ll \tilde{\sigma}(r)$ for all $r > 0$: Since T is bounded we can pick $s \gg \sup T(\mathbb{R}^n_+)$, so that clearly $s \in \Omega(T)$. Then with σ given by the previous result, we define

$$\tilde{\sigma}(r) := \begin{cases} \sigma(r) & \text{if } r \le 1, \\ r \cdot s & \text{otherwise,} \end{cases}$$

to obtain the desired unbounded path.

Following a similar line of argument as in the proof of Proposition 5.2, we obtain a corresponding result for the set Ψ under essentially weaker assumptions on the monotone operator.

Corollary 5.3. *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ a monotone operator satisfying* (2.1). *Then for all $x \in \Psi$ there exists a path $\sigma : [0,1] \to \Psi$ such that $\sigma(0) = 0$, $\sigma(1) = x$ and the component functions of σ are continuous and nondecreasing. Moreover, σ can be chosen to be piecewise linear on* $(0,1]$. *As a consequence, Ψ is pathwise connected.*

The difficulty now is to extend σ in the unbounded direction, so that each component function of the parametrization is unbounded, when T itself is unbounded. To do so, we develop a few technical tools.

Given a monotone operator $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, define the set

$$\Psi_\infty(T) := \bigcap_{k=0}^{\infty} T^k(\Psi)\,. \tag{5.1}$$

Clearly we have $0 \in \Psi_\infty(T) \subset \Psi(T)$.

Now the idea to extend the path given by Proposition 5.2 in the unbounded direction is roughly as follows. If the set Ψ_∞ is radially unbounded and a subset of $\Omega \cup \{0\}$, then pick the initial point $s \in \Omega \cap \Psi_\infty$ in Proposition 5.2 such that its pre-images in Ψ_∞ form a radially unbounded, strictly increasing sequence. Then linearly interpolate these sequence points to get the desired path parametrization. Unfortunately, in practice it is a little more complicated than that, so that the rest of this section is devoted to the technicalities.

Proposition 5.4. *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a monotone operator satisfying* (2.1)*. Assume that*

$$\|T(x^k)\| \to \infty \text{ for any sequence } \|x^k\| \to \infty\,. \tag{5.2}$$

Then $\Psi_\infty(T) \cap S_r \neq \emptyset$ for all $r \geq 0$. In particular, $\Psi_\infty(T)$ is unbounded.

Proof. By Corollary 5.3 the set Ψ is pathwise connected, contains 0, and by Theorem 3.3 Ω and hence Ψ is unbounded. It is easy to see that for any $l \geq 0$ the operator T^l satisfies (5.2) if T does. For every $k \geq 0$ we have $0 \in T^k(\Psi)$, hence $0 \in \Psi_\infty \cap S_0$. Now let $r > 0$, $k \geq 0$, and consider $M = M(k,r) = T^k(\Psi(T)) \cap S_r$. The operator T^k satisfies (5.2) and Ψ is unbounded, so there exists an $s \in \Psi(T)$ such that $\|T^k(s)\| \geq r$. Since $\Psi(T)$ is pathwise connected there exists a path connecting 0 and s in $\Psi(T)$. By continuity of T^k this leads to a path connecting 0 and $T^k(s)$ in $T^k(\Psi(T))$. Clearly this path intersects M, so M is nonempty. It is also clear that M is bounded. Now we show that M is closed: Let $\{v^l\}_{l\in\mathbb{N}}$ be a Cauchy sequence in M. Thus $v^l \to v$ for some $v \in \mathbb{R}^n_+$ as $l \to \infty$. Since $\{v^l\}_{l\in\mathbb{N}} \subset T^k(\Psi(T))$ there exist $w^l \in \Psi(T)$ such that $v^l = T^k(w^l)$. Assume the sequence $\{w^l\}$ was unbounded. Then by (5.2) also $\{v^l\}$ had to be unbounded, in contradiction to $v^l \to v$. Hence $\{w^l\}_{l\in\mathbb{N}}$ is a bounded sequence in $\Psi(T) \subset \mathbb{R}^n_+$, so it has a convergent subsequence, which for notational simplicity we again denote by $\{w^l\}$. Denote its limit by w, which is an element of $\Psi(T)$, since $\Psi(T)$ is closed. So by continuity of T^k we conclude $v = T^k(w)$, proving that M is closed. Together with boundedness we have that M is compact for any choice of k and r. By Lemma 3.5.1 the sets $T^k(\Psi)$ are nested. So the sets $M^k_r = M(k,r)$, $k \geq 0$, are nested and compact for fixed $r > 0$. A decreasing family of compact sets has nonempty intersection (c.f., [10, Cantor Theorem, p.171] or [13, Theorem 3.5.9, p.170]), so for any $r > 0$ the set $M_r = \bigcap_k M^k_r$ is nonempty. This proves the claim. □

The first result towards the path construction problem assumes strong connectedness of the underlying graph.

Theorem 5.5. *Let $\Gamma \in \mathcal{G}^{n\times n}$ be irreducible, $\mu \in \mathrm{MAF}_n^n$ satisfy (M3), and Γ_μ satisfy* (2.1)*. Then there exists $\sigma \in \mathcal{K}_\infty^n$ such that $\sigma(r) \in \Omega(\Gamma_\mu)$ for all $r > 0$. Moreover, σ can be chosen to be piecewise linear on* $(0,\infty)$.

The proof comprises the following steps: First we construct a $\mathcal{K}_\infty$ function $\varphi > \mathrm{id}$ so that for $D = \mathrm{diag}(\varphi)$ we have $\Gamma_\mu \circ D \not\geq \mathrm{id}$, see Proposition 5.8. In other words, an operator of the form Γ_μ with irreducible Γ satisfying (2.1) satisfies this condition in a robust sense; the entries of Γ can all be slightly increased by a function $\varphi > \mathrm{id}$.

Then we construct a monotone (but not necessarily strictly increasing) sequence $\{s^k\}_{k\geq 0}$ in $\Psi(\Gamma_\mu \circ D)$ so that each component sequence is unbounded. Finally we use the little extra space provided by D in the set $\Omega(\Gamma_\mu) \supset \Omega(\Gamma_\mu \circ D)$ to obtain a strictly increasing sequence $\{\tilde{s}^k\}_{k\geq 0}$ in $\Omega(\Gamma_\mu)$ which we can linearly interpolate to obtain the desired $\mathcal{K}_\infty$-path in $\Omega \cup \{0\}$.

An important implication of the prerequisites of Theorem 5.5 is the following application of Proposition 5.4 to operators Γ_μ with irreducible Γ and μ satisfying (M3).

Proposition 5.6. *Let $\Gamma \in \mathcal{G}^{n\times n}$ be irreducible, $\mu \in \mathrm{MAF}_n^n$ satisfy (M3), and Γ_μ satisfy* (2.1)*. Then $\Psi_\infty(\Gamma_\mu) \cap S_r \neq \emptyset$ for all $r \geq 0$ and $\Psi_\infty(\Gamma_\mu)$ is radially unbounded.*

Proof. Due to irreducibility of Γ and the extra prerequisite on μ the following property holds: For any pair $1 \leq i,j \leq n$ there exists a $k \geq 1$ such that

$$r \mapsto (\Gamma_\mu^k(re_j))_i \tag{5.3}$$

is an unbounded function, where e_j is the jth unit vector. Hence Γ_μ and also all powers Γ_μ^l of Γ_μ, $l \geq 1$, satisfy (5.2).

By Proposition 5.4 the set $\Psi_\infty(\Gamma_\mu)$ is unbounded, so we may pick an unbounded sequence $\{s^k\} \subset \Psi_\infty(\Gamma_\mu)$, such that for some index i we have $s_i^k \to \infty$ as $k \to \infty$. By definition of Ψ_∞ we have $\Gamma_\mu^l(\Psi_\infty) \subset \Psi_\infty$ and hence $\Gamma_\mu^l(\{s^k\}_k) \subset \Psi_\infty$ for all $l \geq 0$. Using (5.3) we find that for all j there exists an l such that $[\Gamma_\mu^l(\{s^k\})]_j$ is unbounded. Since for every $l,k \geq 0$, $\Gamma_\mu^l(s^k) \leq s^k$, it follows that the sequence $\{s^k\}$ must be unbounded in every component. Hence the set Ψ_∞ must be unbounded in every coordinate direction, i.e., Ψ_∞ is radially unbounded. The other assertion follows from Proposition 5.4. □

We proceed with the construction of the robustness operator D. Before, we have a little technical lemma:

Lemma 5.7. *Let $\Gamma \in \mathcal{G}^{n\times n}$ be irreducible, $\mu \in \mathrm{MAF}_n^n$ satisfy (M3), and Γ_μ satisfy* (2.1)*. Then Ω is radially unbounded.*

Proof. We show that $\overline{\Omega}$ is radially unbounded. Since Ω is open, this implies the claim. By Lemma 2.7 the monotone operator Γ_μ is strictly increasing. Observe that for any $k \geq 0$, $\Gamma_\mu^k(\Psi) \supset \Gamma_\mu^k(\Omega) \subset \Omega$, so for $r > 0$, the intersection $S_r \cap \Gamma_\mu^k(\Psi) \cap \Omega$ is nonempty, hence contains a point s_k^r. For fixed r, the sequence $\{s_k^r\}$ is bounded and therefore contains an accumulation point in $S_r \cap \overline{\Omega} \cap \Psi_\infty$. The set Ψ_∞ is radially unbounded by the previous result, so must be the set $\overline{\Omega}$. □

Proposition 5.8. *Let $\Gamma \in \mathcal{G}^{n\times n}$ be irreducible, $\mu \in \mathrm{MAF}_n^n$ satisfy (M3), and Γ_μ satisfy* (2.1). *Then there exists a diagonal operator $D = \operatorname{diag}(\varphi)$, $\varphi \in \mathcal{K}_\infty$, $\varphi > id$, such that $(\Gamma_\mu \circ D)(x) \not\geq x$ for all $x \neq 0$.*

Proof. By Lemma 2.7 Γ_μ is strictly increasing. Let the family of compact sets $\{I_K\}_{K=-\infty}^{\infty}$ be a covering of $(0,\infty)$ such that $I_K \cap I_{K+1}$ consists of exactly one point. These sets will be used to piecewise construct an operator $D = \operatorname{diag}(\rho)$, $\rho \in \mathcal{K}_\infty$, $\rho > \mathrm{id}$, such that $\Gamma_\mu \circ D \not\geq \mathrm{id}$.

For a fixed K, by employing Proposition 5.4 and Theorem 3.3, we pick a point $s^0 = s^0(K) \in \Omega$ satisfying the following properties: $\|s^0\| > \|s^1\| > \max I_K > \|s^2\|$, where s^k is defined by

$$s^{k+1} = \Gamma_\mu(s^k), \quad k \geq 0. \tag{5.4}$$

As in the proof of Proposition 5.2 the sequence $\{s^k\}_{k\geq 0}$ converges to zero. Moreover, by Lemma 3.5 we have $s^k \in \Omega$ for all $k \geq 0$. By Lemma 5.7 the set Ω is radially unbounded, so we may assume that the sequence $\{s_K^0\}_{K=-\infty}^{\infty}$ is also radially unbounded. The path segments defined by

$$p_k(\lambda) := \Gamma_\mu^k\big((1-\lambda)s^1 + \lambda s^0\big), \quad k \geq 0,\, \lambda \in [0,1),$$

piecewise parametrize a path in Ω, connecting an arbitrarily small neighborhood of 0 and s^0. They are all disjoint, satisfy the monotonicity conditions

$$p_{k+1}(\lambda) = \Gamma_\mu(p_k(\lambda)) \ll p_k(\lambda)$$
$$p_k(\lambda) \ll p_k(\tilde{\lambda}) \text{ if } \lambda < \tilde{\lambda},$$

and for each $r \in (0, \|s^0\|)$ there is a unique pair $(k,\lambda) = (k_r, \lambda_r)$ such that $|p_k(\lambda)| = r$. Along the path defined by the path segments $\{p_k\}$ define an operator D_K by

$$D_K(p_k(\lambda)) = \begin{cases} p_{k-1}(\lambda - \frac{1}{2}) & \text{if } \lambda \geq \frac{1}{2}, \\ p_k(\lambda + \frac{1}{2}) & \text{if } \lambda < \frac{1}{2}. \end{cases} \tag{5.5}$$

This operator satisfies $D_K(p_k(\lambda)) = D_K(\Gamma_\mu(p_{k-1}(\lambda))) \ll p_{k-1}(\lambda)$ for all $k \geq 1$. In particular we have

$$(D_K \circ \Gamma_\mu)^{2l}(p_k(\lambda)) \leq p_{k+l}(\lambda) \longrightarrow 0 \quad \text{as} \quad l \to \infty.$$

For $r \in I_K$ define

$$d_K(r) := \min_i \frac{D_K(p_{k(K,r)}(\lambda(K,r)))_i}{(p_{k(K,r)}(\lambda(K,r)))_i}.$$

Clearly $d_K(r) > 1$ for $r \in I_K$. Hence we can find a continuously differentiable, positive definite function $\tilde{\varphi} : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any K, $\tilde{\varphi}(r) < d_K(r)$ for all $r \in I_K$, and such that $\tilde{\varphi}'(r) < 1$ for all $r > 0$. Now let $\varphi = \mathrm{id} + \tilde{\varphi}$, then clearly φ is of class $\mathcal{K}_\infty$.

The operator $D = \mathrm{diag}(\varphi)$ satisfies, for any K along the path segments $p_k = p_k(K)$, that $p_k(\lambda) \ll D(p_k(\lambda)) \ll D_K(p_k(\lambda))$. Hence for any $x \in \mathbb{R}^n_+$ we can find some K, k, λ such that $p_k(\lambda) = p_k(\lambda, K) \geq x$. By monotonicity we have

$$(D \circ \Gamma_\mu)^l(x) \leq (D_K \circ \Gamma_\mu)^l(p_k(\lambda)) \ll p_{k+\lfloor \frac{l}{2} \rfloor}(\lambda) \longrightarrow 0 \quad \text{as} \quad l \to \infty. \tag{5.6}$$

By Proposition 4.1 this implies $D \circ \Gamma_\mu \not\geq \mathrm{id}$. From Lemma 2.6 $(\Gamma_\mu \circ D)(x) \not\geq x$ for $x \neq 0$ follows. □

Proof of Theorem 5.5. Let $D = \mathrm{diag}(\varphi)$, $\varphi \in \mathcal{K}_\infty$, $\varphi > \mathrm{id}$, such that $(\Gamma_\mu \circ D)(x) \not\geq x$ for all $x \neq 0$, be given by Proposition 5.8. By Lemma 2.7 Γ_μ is strictly increasing, and so is $\Gamma_\mu \circ D$. Now we construct a nondecreasing and unbounded sequence $\{s^k\}$ in $\Psi(\Gamma_\mu \circ D)$:

Let $T := \Gamma_\mu \circ D$. As just observed we have $T(s) \not\geq s$ for all $s \neq 0$. By Proposition 5.6 the set $\Psi_\infty(T)$ is radially unbounded. By Lemma 3.6 we have $\Psi_\infty(T) \cap \{s \in \mathbb{R}^n_+ : s \gg 0\} \subset \Psi(T) \cap \{s \in \mathbb{R}^n_+ : s \gg 0\} \subset \Omega(\Gamma_\mu)$. So since $\Psi_\infty(T)$ is radially unbounded, we may pick a point $s^0 \in \Psi_\infty(T) \cap \{s \in \mathbb{R}^n_+ : s \gg 0\}$. Now define a sequence $\{s^k\}_{k\geq 0}$ by

$$s^{k+1} \in T^{-1}(s^k) \cap \Psi_\infty(T)$$

for $k \geq 0$. This is possible, since $\Psi_\infty(T)$ is backward invariant under T.

The so defined sequence $\{s^k\}$ satisfies $s^k < s^{k+1}$ by definition. We claim that it is also unbounded, and, by the same argument as in the proof of Proposition 5.6, unbounded in every component: To this end assume that it is bounded. Then by monotonicity there exists a limit $s^* = \lim_{k\to\infty} s^k$. By continuity of T and since $s^k = T(s^{k+1})$ we have

$$s^* = \lim_{k\to\infty} s^k = \lim_{k\to\infty} T(s^{k+1}) = T\left(\lim_{k\to\infty} s^{k+1}\right) = T(s^*)$$

contradicting $T(s) \not\geq s$ for all $s \neq 0$. Hence the sequence $\{s^k\}$ must be unbounded.

Now the sequence $\{s^k\}$ lives in $\Omega(\Gamma_\mu)$, but it may not be strictly increasing, as we only know $s^k < s^{k+1}$ for all $k \geq 0$. We define a strictly increasing sequence $\{\tilde{s}^k\}$ as follows: By Lemma 2.5 for any $k \geq 0$ we may factorize $D = D_1^{(k)} \circ D_2^{(k)}$ in such a way that $D_2^{(k)}(s) \ll D_2^{(k+1)}(s)$ for all $k \geq 0$ and all $s \gg 0$. Using this factorization we define

$$\tilde{s}^k := D_2^{(k)}(s^k)$$

for all $k \geq 0$. By the definition of $D_2^{(k)}$, this sequence is clearly strictly increasing and inherits from $\{s^k\}$ the unboundedness in all components. We claim that $\{\tilde{s}^k\} \subset \Omega(\Gamma_\mu)$: This follows from

$$\tilde{s}^k \gg s^k \geq \Gamma_\mu \circ D(s^k) = \Gamma_\mu \circ D_1^{(k)} \circ D_2^{(k)}(s^k) = \Gamma_\mu \circ D_1^{(k)}(\tilde{s}^k) \gg \Gamma_\mu(\tilde{s}^k).$$

Now we prove that for $\lambda \in (0,1)$ we have $(1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1} \in \Omega(\Gamma_\mu)$. Clearly

$$\tilde{s}^k \ll (1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1} \ll \tilde{s}^{k+1}$$

and application of the strictly increasing operator Γ_μ yields

$$\begin{aligned}\Gamma_\mu((1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1}) &\ll \Gamma_\mu(\tilde{s}^{k+1}) \\ &= \Gamma_\mu \circ D_2^{(k+1)}(s^{k+1}) \ll \Gamma_\mu \circ D_1^{(k+1)} \circ D_2^{(k+1)}(s^{k+1}) \\ &= s^k \ll \tilde{s}^k \ll (1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1}.\end{aligned}$$

Hence $(1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1} \in \Omega(\Gamma_\mu)$.

Now we may define σ as a parametrization of the linear interpolation of the points $\{\tilde{s}^k\}_{k\geq 0}$ in the unbounded direction and utilize the construction from Proposition 5.2 for the other direction.

Clearly this function σ has component functions of class $\mathcal{K}_\infty$ and is piecewise linear on every compact interval contained in $(0,\infty)$. □

In the following example it turns out that in full generality condition (2.1) is not sufficient for the existence of a path in the set Ω parametrized by $\mathcal{K}_\infty$ functions.

Example 5.9. *Let $\mu = \Sigma \in \mathrm{MAF}_2^2$ and $\Gamma \in \mathcal{G}^{2\times 2}$ be given by*

$$\Gamma = \begin{pmatrix} \gamma_{11} & id \\ 0 & \gamma_{22} \end{pmatrix},$$

where $\gamma_{11}(t) = \gamma_{22}(t) = t(1-e^{-t})$ are of class $\mathcal{K}_\infty$. Clearly Γ is reducible and $\Gamma_\mu(s) \not\geq s$ for all $s \neq 0$ (just consider the cases $s \gg 0, s = (s_1, 0)^T > 0$, and $s = (0, s_2)^T > 0$ separately).

Let us consider $\Omega_1 = \{s \in \mathbb{R}^2 : s_1(1-e^{-s_1}) + s_2 < s_1\} = \{s \in \mathbb{R}^2 : s_2 < s_1 e^{-s_1}\}$ and $\Omega_2 = \{s \in \mathbb{R}^2 : s_2(1-e^{-s_2}) < s_2\} = \{s \in \mathbb{R}^2 : s_2 \neq 0\}$. The intersection $\Omega = \Omega_1 \cap \Omega_2$ is given by

$$\Omega = \{s \in \mathbb{R}^2 : s_2 < s_1 e^{-s_1}, s_2 \neq 0\}.$$

This set is clearly not unbounded in the s_2-direction, as its vertical cross sections for large s_1 asymptotically are arbitrarily close to the s_1-axis.

By strengthening condition (2.1) by a "robustness" operator D, we can ensure that the set Ω is radially unbounded, so that it can accommodate the desired path.

Theorem 5.10. *Let $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$ satisfy (M3),(M4). Assume there exists a $\rho \in \mathcal{K}_\infty$ such that for $D = \mathrm{diag}(id+\rho)$ the operator $D \circ \Gamma_\mu$ satisfies* (2.1). *Then there exists a $\tilde{D} = \mathrm{diag}(id+\tilde{\rho})$, $\tilde{\rho} \in \mathcal{K}_\infty$, and a $\mathcal{K}_\infty^n$-path $\sigma : \mathbb{R}_+ \to \mathbb{R}_+^n$ such that $\Gamma_\mu \circ \tilde{D}(\sigma(r)) \ll \sigma(r)$ for all $r > 0$. Moreover, σ can be chosen to be piecewise linear on $(0,\infty)$.*

Proof. By Lemma 2.5 we may assume that there exist $D_i = \mathrm{diag}(\mathrm{id}+\rho_i)$, $\rho_i \in \mathcal{K}_\infty$, $i = 1, 2$, such that $D_1 \circ D_2 \circ \Gamma_\mu(s) \not\geq s$ for all $s \neq 0$. We may also assume that Γ_μ is in upper triangular block form with irreducible and 1×1-zero blocks on the diagonal, as detailed in Section 2.9. We construct σ by induction over the number of irreducible and 1×1-zero blocks on the diagonal of Γ. The argument is as follows.

Induction start. If Γ is irreducible or a 1×1-zero-block, the claim follows from Theorem 5.5 or by the discussion following Proposition 5.2.

We may further assume that the function σ given by these results is piecewise linear on $(0, \infty)$.

Induction step. A reducible matrix in upper triangular block form with a total of $k+1$ irreducible and zero 1×1-blocks B_l, $l = 1, \dots, k+1$, on the diagonal can be considered as a block matrix with only two diagonal blocks, the first of which possibly reducible (if $k > 1$) and the other one zero/irreducible. So we may assume that Γ is reducible into

$$\Gamma = \left(\begin{array}{ccc|c} B_1 & * & * & * \\ 0 & \ddots & * & * \\ 0 & 0 & B_k & * \\ \hline 0 & 0 & 0 & B_{k+1} \end{array}\right) = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix}$$

for some $\Gamma_{11} \in \mathcal{G}^{n_1 \times n_1}$, $\Gamma_{12} \in \mathcal{G}^{n_1 \times n_2}$, and $\Gamma_{22} \in \mathcal{G}^{n_2 \times n_2}$ for some $n_1, n_2 \geq 1$, $n_1 + n_2 = n$, so that $\Gamma_{22} = B_{k+1}$ is irreducible or a 1×1-zero block. We may further assume that we have (with D_2 adapted according to the dimensions)

$$D_1 \circ D_2 \circ (\Gamma_{ii})_{\mu_i}(x) \not\geq x$$

for all $x \in \mathbb{R}_+^{n_i}$, $i = 1, 2$, with $\mu = (\mu_1, \mu_2)$.

Induction hypothesis: There exists a function $\sigma_1 \in \mathcal{K}_\infty^{n_1}$ such that

$$D_2 \circ (\Gamma_{11})_{\mu_1}(\sigma_1(r)) \ll \sigma_1(r) \tag{5.7}$$

for all $r > 0$. Now, since Γ_{22} is irreducible or a 1×1-zero-block, by Theorem 5.5 or by the discussion following Proposition 5.2 there also exists a function $\sigma_2 \in \mathcal{K}_\infty^{n_2}$ satisfying

$$D_1 \circ D_2 \circ (\Gamma_{22})_{\mu_2}(\sigma_2(r)) \ll \sigma_2(r) \tag{5.8}$$

for all $r > 0$. There is no restriction in assuming that both σ_1 and σ_2 are piecewise linear on compact intervals in $(0, \infty)$. By Lemma 2.5 we may again alter D such that (5.7) and (5.8) can be rewritten as

$$D^2 \circ (\Gamma_{ii})_{\mu_i}(\sigma_i(r)) \ll \sigma_i(r)$$

or equivalently

$$D \circ (\Gamma_{ii})_{\mu_i}(\sigma_i(r)) \ll D^{-1}(\sigma_i(r))$$

for all $r > 0$. By Lemma 2.4 D^{-1} is of the form $\operatorname{diag}(\mathrm{id} - \tilde{\rho})$ for some $\tilde{\rho} \in \mathcal{K}_\infty$. This is exactly the point were it is essential that we have D of the form $\operatorname{diag}(\mathrm{id} + \rho)$ instead of $\operatorname{diag}(\rho)$. Write $R = \operatorname{diag}(\tilde{\rho})$. Define a function $\tau : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$,

$$\tau(r) := \sup\{\xi \in \mathbb{R}_+ : D \circ (\Gamma_{12})_\mu(\sigma_2(\xi)) \leq R(\sigma_1(r))\}.$$

If $\Gamma_{12} \neq 0$ then this function is strictly increasing and as $r \to \infty$ we have $\tau(r) \to \infty$. In addition τ satisfies $\tau(0) = 0$ and $\tau(r) > 0$ for all $r > 0$. Clearly τ is continuous and therefore of class $\mathcal{K}_\infty$. If $\Gamma_{12} = 0$ we can bound τ from below by a function of class $\mathcal{K}_\infty$. For simplicity we denote this function again by τ.

Now bound τ from below with a function $\hat{\tau}$ of class $\mathcal{K}_\infty$ that is piecewise linear on any compact interval in $(0,\infty)$ — this is possible for any $\mathcal{K}_\infty$ function τ.

Hence for all $r \geq 0$ we have

$$D \circ (\Gamma_{12})_\mu(\sigma_2(\hat{\tau}(r))) \leq R(\sigma_1(r)).$$

Now consider

$$\begin{aligned}
\pi_{I_1} \circ D \circ \Gamma_\mu((\sigma_1(r), \sigma_2 \circ \hat{\tau}(r))^T) &\leq D \circ (\Gamma_{11})_\mu(\sigma_1(r)) \\
&\quad + D \circ (\Gamma_{12})_\mu \circ \sigma_2 \circ \hat{\tau}(r) \qquad (5.9) \\
&\ll D^{-1}(\sigma_1(r)) + D \circ (\Gamma_{12})_\mu \circ \sigma_2 \circ \hat{\tau}(r) \\
&= \sigma_1(r) - R(\sigma_1(r)) \\
&\quad + D \circ (\Gamma_{12})_\mu \circ \sigma_2 \circ \hat{\tau}(r) \\
&\leq \sigma_1(r)\,,
\end{aligned}$$

where in (5.9) we have used (M4). Hence we have established

$$D \circ \Gamma_\mu\left(\begin{pmatrix} \sigma_1(r) \\ \sigma_2 \circ \hat{\tau}(r) \end{pmatrix}\right) \ll \begin{pmatrix} \sigma_1(r) \\ \sigma_2 \circ \hat{\tau}(r) \end{pmatrix} =: \sigma(r),$$

implying that $\sigma(r) \in \Omega(D \circ \Gamma_\mu) \subset \Omega(\Gamma_\mu)$ for all $r > 0$. By construction σ is piecewise linear on every compact interval contained in $(0,\infty)$. This completes the induction step. □

6. Solutions to inequalities

We come back to the first of our problems, which, as it turns out, is intrinsically related to the no-joint-increase condition (2.1) for the operator $\Gamma_\mu : \mathbb{R}^n_+ \to \mathbb{R}^n_+$.

Theorem 6.1. *Let $\Gamma \in \mathcal{G}^{n\times n}$ and let $\mu \in \mathrm{MAF}^n_n$ satisfy (M4). Assume there exists a $D = \mathrm{diag}(id + \rho)$, $\rho \in \mathcal{K}_\infty$, such that $D \circ \Gamma_\mu(x) \not\geq x$ for all $x \neq 0$. Then there exists a $\varphi \in \mathcal{K}_\infty$ such that for all $w, v \in \mathbb{R}^n_+$,*

$$(id - \Gamma_\mu)(w) \leq v \qquad (6.1)$$

implies $\|w\| \leq \varphi(\|v\|)$.

In fact, this result has been proved for the special case $\mu_i(x) = \sum_j x_j$ already in [2], using a slightly different notation. The differences in the proof to the more general $\mu \in \mathrm{MAF}^n_n$ are only marginal.

To see that the no-joint-increase condition (2.1) itself is not a restrictive assumption for Theorem 6.1 to hold, consider the following:

Example 6.2. *Assume $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}^n_n$ and that there exists a function $\varphi \in \mathcal{K}_\infty$ such that for all $w, v \in \mathbb{R}^n_+$, inequality (6.1) implies $\|w\| \leq \varphi(\|v\|)$. Then $\Gamma_\mu(x) \not\geq x$ for all $x \neq 0$: Indeed, if we assume there exists an $x \in \mathbb{R}^n_+$ such that $\Gamma_\mu(x) \geq x$, then we can rewrite this inequality to obtain*

$$(id - \Gamma_\mu)(x) \leq 0$$

which is just (6.1) *with* $v = 0$. *This implies* $\|x\| \le \varphi(0)$, *hence it must be* $x = 0$. *In other words,* $\Gamma_\mu(x) \ngeq x$ *if* $x \neq 0$.

This consideration raises the question, whether or not the operator D is really needed in the assumptions of Theorem 6.1. In general the operator D is essential, if the monotone inequality is stated as in (6.1), as the following example demonstrates.

Example 6.3. *Consider*

$$\Gamma = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}$$

with $\gamma(t) = t(1 - e^{-t})$ *of class* $\mathcal{K}_\infty$ *and* $\mu = \Sigma$ *or* $\mu = \oplus$. *We have* $\Gamma_\mu(x) \ngeq x$ *for any* $x \neq 0$.

We show that there exists no function $\varphi \in \mathcal{K}_\infty$ *such that the monotone inequality*

$$(id - \Gamma_\mu)(w) \le v \tag{6.2}$$

for $w = (w_1, w_2)^T$, $v = (v_1, v_2)^T$ *implies* $\|w\| \le \varphi(\|v\|)$. *The monotone inequality* (6.2) *applied to* $w = (w_1, w_1)$ *and* $v = (v_1, v_1)$ *leads to*

$$w_1 e^{-w_1} = w_1 - w_1(1 - e^{-w_1}) \le v_1 \tag{6.3}$$

Now take $v_1 = 1$, *then inequality* (6.3) *is satisfied for any* $w_1 \ge 0$. *If there existed a function* $\varphi \in \mathcal{K}_\infty$ *such that* (6.2) *implied* $\|w\| \le \varphi(\|v\|)$, *we would have* $\|w\| \le \varphi(\|(1,1)^T\|)$ *for any choice of* $w = (w_1, w_1)^T$. *Clearly such a* $\mathcal{K}_\infty$*-function* φ *cannot exist.*

Nevertheless, there exists a special case, where the operator D is not needed.

Theorem 6.4. *Let* $\Gamma \in \mathcal{G}^{n\times n}$. *Then the following are equivalent:*

1. $\Gamma_\oplus(x) \ngeq x$ *for all* $x \neq 0$.
2. *All cycles in* Γ *are contractions, i.e.,* $\gamma_{i_0 i_1} \circ \gamma_{i_1 i 2} \circ \ldots \circ \gamma_{i_k i_0} < id$.
3. *There exists a function* $\varphi \in \mathcal{K}_\infty$ *such that for any* $b \in \mathbb{R}^n_+$ *the solutions* x *of the inequality*
$$x \le \Gamma_\oplus(x) \oplus b \tag{6.4}$$
satisfy $\|x\| \le \varphi(\|b\|)$.
4. *The origin is GAS with respect to*
$$x^{k+1} = \Gamma_\oplus(x^k)\,. \tag{6.5}$$
5. *There exists* $\sigma \in \mathcal{K}^n_\infty$ *such that* $\Gamma_\oplus(\sigma(r)) \ll \sigma(r)$ *for all* $r > 0$*; moreover,* σ *may be assumed to be piecewise linear on* $(0, \infty)$.

At first sight it may seem that this result contradicts Example 6.3. The important difference here is the formulation of the monotone inequality (6.4), which is not equivalent to

$$(x - \Gamma_\oplus(x)) \le b. \tag{6.6}$$

In fact, by Example 6.3 we see that (6.6) is strictly weaker than (6.4).

If the functions γ_{ij} are all linear, then $\Gamma_\oplus$ is a max linear operator, see [12]. Note that this is not the same as a max-plus linear operator. The cycle condition is an equivalent formulation of saying that the max spectral radius has to be less than one. In general $\mu(A) \leq \rho(A)$, where $\mu(A)$ is the maximal cycle geometric mean of a nonnegative matrix A (corresponding to the cycle condition), and $\rho(A)$ is the usual spectral radius of A, cf. [12].

An implication of the form 2 to 3 in Theorem 6.4 also has been proved in [14] and the equivalence of the two has been proved in [17], where also the connection to assertion 4 has been observed.

Proof of Theorem 6.4. Here we prove the following implications: $1 \implies 2 \implies 1$ and (1 and 4) $\implies 3 \implies 1$. The other implications can be derived as follows: $4 \implies 1$ is given by Proposition 4.1. Clearly $5 \implies 4$ by Lemma 5.1. If Γ is irreducible, then Theorem 5.5 gives $1 \implies 5$. If Γ is reducible, the proof for $1 \implies 5$ is essentially the same as the one for Theorem 5.10, with the only difference, that due to the maximization the operator D is not needed.

1 $\implies$ 2: The no-joint-increase condition implies the cycle condition: We argue by contradiction. Suppose there is a cycle such that for some $r \in \mathbb{R}_+$ we have $\gamma_{i_0 i_1} \circ \gamma_{i_1 i_2} \circ \ldots \circ \gamma_{i_{k-1} i_k}(r) \geq r$ with $i_0 = i_k$. Then $\pi_{i_0}(\Gamma_\oplus^k(r \cdot e_{i_0})) \geq \gamma_{i_0 i_1} \circ \gamma_{i_1 i_2} \circ \ldots \circ \gamma_{i_{k-1} i_k}(r) \geq r$ and hence $\Gamma_\oplus^k(r \cdot e_{i_0}) \geq r \cdot e_{i_0}$. But this contradicts $\Gamma_\oplus^k(x) \not\geq x$ for all $x \neq 0$, which is necessary for the no-joint-increase condition to hold, cf. Lemma 2.1, so the no-joint-increase condition cannot hold.

2 $\implies$ 1: We argue again by contradiction. Assume the cycle condition is true while there exists an $x > 0$, such that $\Gamma_\oplus(x) \geq x$. Choose i such that x_i is maximal, i.e., $x_i = \max_j x_j$. By monotonicity of $\Gamma_\oplus$ we have $x \leq \Gamma_\oplus^k(x)$ for all $k \geq 1$ and hence

$$\begin{aligned} 0 < x_i &\leq \pi_i \circ \Gamma_\oplus^k(x) \\ &= \max_{l_1,\ldots,l_{k-1},j} \gamma_{il_1} \circ \ldots \circ \gamma_{l_{k-1}j}(x_j) \\ &\leq \max_{l_1,\ldots,l_{k-1},j} \gamma_{il_1} \circ \ldots \circ \gamma_{l_{k-1}j}(x_i). \end{aligned} \tag{6.7}$$

Now the number of functions γ_{lm} in each term in the maximization in (6.7) that are not part of a cycle is bounded by $2(n-1)$. To see this, note that a loop-free path of maximal length in a graph with n vertices consists of at most $n-1$ edges. This implies that the composition $\gamma_{il_1} \circ \ldots \circ \gamma_{l_{k-1}j}$ in (6.7) can be written as

$$\underbrace{\gamma_{il_1} \circ \ldots \circ \gamma_{l_{\lambda-1}l_\lambda}}_{\text{without loops}} \circ \underbrace{\gamma_{l_\lambda l_{\lambda+1}} \circ \ldots \circ \gamma_{l_{\nu-1}l_\nu}}_{\text{only loops}} \circ \underbrace{\gamma_{l_\nu l_{\nu+1}} \circ \ldots \circ \gamma_{l_{k-1}j}}_{\text{without loops}} \tag{6.8}$$

where the first and the last part are possibly of zero length, but each at most of length $n-1$ each.

As k tends to infinity, the number of cycles in (6.8), respectively in the right hand side (6.7) increases. Each such cycle is less than the identity, hence for $k \to \infty$ the right hand side (6.7) tends to zero, contradicting $x_i > 0$.

1 and 4 $\implies$ 3: Assume $\Gamma_\oplus(x) \not\geq x$ for all $x \neq 0$. Consider

$$\begin{aligned} x &\leq \Gamma_\oplus(x) \oplus b \\ &\leq \Gamma_\oplus(\Gamma_\oplus(x) \oplus b) \oplus b \\ &= \Gamma_\oplus(\Gamma_\oplus(x)) \oplus \Gamma_\oplus(b) \oplus b \\ &\leq \ldots \leq \Gamma_\oplus^k(x) \oplus \bigoplus_{l=0}^{k-1} \Gamma_\oplus^l(b) \end{aligned} \tag{6.9}$$

for all $k \geq 1$. By assertion 4 we know that $O(s) = \{\Gamma_\oplus^k(s)\}_{k\geq 0}$ is bounded for every $s \in \mathbb{R}_+^n$ and $\Gamma_\oplus^k(s) \to 0$ as $k \to \infty$. Hence the term $\Gamma_\oplus^k(x)$ tends to zero as $k \to \infty$ and for fixed b the term $\bigoplus_{l=0}^{k-1} \Gamma_\oplus^l(b)$ is bounded in k. We obtain from (6.9) for $k \to \infty$ that $x \leq \bigoplus_{l=0}^{\infty} \Gamma_\oplus^l(b)$, so that $\|x\|$ is clearly bounded by some $\mathcal{K}_\infty$-function φ of $\|b\|$: To this end note that the supremum $\bigoplus_{l=0}^{\infty} \Gamma_\oplus^l(b)$ is actually attained in finitely many steps, i.e., for some $k_0 \geq 0$ such that $\bigoplus_{l=0}^{\infty} \Gamma_\oplus^l(b) = \bigoplus_{l=0}^{k_0} \Gamma_\oplus^l(b)$. This is a consequence of the fact that $\{\Gamma_\oplus^l(b)\}_{l\geq 0}$ is a null sequence, and the supremum over a null sequence in $\mathbb{R}_+^n$ with respect to the standard partial order is attained after finitely many steps.

3 $\implies$ 1: See Example 6.2. □

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References

[1] A. Berman and R. J. Plemmons. *Nonnegative matrices in the mathematical sciences.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1979.

[2] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS small gain theorem for general networks. *Mathematics of Control, Signals, and Systems (MCSS)*, 19(2):93–122, 2007.

[3] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. January 2009, submitted.

[4] W. Hahn. *Theorie und Anwendung der direkten Methode von Ljapunov.* Ergebnisse der Mathematik und ihrer Grenzgebiete N. F., Heft 22. Springer-Verlag, Berlin, 1959.

[5] W. Hahn. *Stability of motion.* Translated from the German manuscript by Arne P. Baartz. Die Grundlehren der mathematischen Wissenschaften, Band 138. Springer-Verlag New York, Inc., New York, 1967.

[6] C. D. Horvath and M. Lassonde. Intersection of sets with n-connected unions. *Proc. Am. Math. Soc.*, 125(4):1209–1214, 1997.

[7] B. Knaster, C. Kuratowski, and S. Mazurkiewicz. Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe. *Fundamenta*, 14:132–137, 1929.

[8] M. A. Krasnosel′skiĭ. *The operator of translation along the trajectories of differential equations.* American Mathematical Society, Providence, R.I., 1968.

[9] M. A. Krasnosel′skiĭ and P. P. Zabreĭko. *Geometrical methods of nonlinear analysis.* Springer-Verlag, Berlin, 1984.

[10] Kazimierz Kuratowski. *Introduction to set theory and topology.* Pergamon Press, 1961.

[11] M. Lassonde. Sur le principe KKM. *C. R. Acad. Sci. Paris Sér. I Math.*, 310(7):573–576, 1990.

[12] Yung-Yih Lur. On the asymptotic stability of nonnegative matrices in max algebra. *Linear Algebra Appl.*, 407:149–161, 2005.

[13] James Raymond Munkres. *Topology: a first course.* Prentice-Hall, 1975.

[14] H. G. Potrykus. *An idempotent-Analytic ISS Small Gain Theorem with Applications to Complex Process Models.* PhD thesis, The University of Texas at Austin, 2002.

[15] H. G. Potrykus, F. Allgöwer, and S. Joe Qin. The character of an idempotent-analytic nonlinear small gain theorem. In *Positive systems (Rome, 2003)*, volume 294 of *Lecture Notes in Control and Inform. Sci.*, pages 361–368. Springer, Berlin, 2003.

[16] B. S. Rüffer. *Monotone Systems, Graphs, and Stability of Large-Scale Interconnected Systems.* PhD thesis, Fachbereich 3, Mathematik und Informatik, Universität Bremen, Germany, 2007. available online at http://nbn-resolving.de/urn:nbn:de:gbv:46-diss000109058.

[17] A. Teel. Input-to-state stability and the nonlinear small gain theorem. Chapter 12 in his lecture notes, courtesy due to private communication, June 2005.

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