Small-gain conditions and the comparison principle

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***Abstract*—The general input-to-state stability (ISS) small-gain condition for networks in a trajectory formulation is shown to be equivalent to the requirement that a discrete-time comparison system induced by the gain matrix of the network is ISS. This leads to a comparison principle, relating input-to-state stability of an artificial discrete-time system to the same stability property of a continuous-time nominal system. As a consequence, general small-gain conditions can now be verified by finding ISS Lyapunov functions.**

***Index Terms*—Stability of NL Systems, discrete-continuous comparison principle, input-to-state stability**

I. Introduction

Small-gain conditions are contraction conditions and are sufficient for the stability of feedback interconnected systems. For nonlinear systems they have been around at least since the 1960s starting with [1]. In the input-to-state stability (ISS) framework such conditions became available in the 1990s with [2], [3]. More recently, these results have been extended to "networked" versions: The interconnection of arbitrarily many ISS systems in arbitrary interconnection yields again an ISS composite system, provided a generalized small-gain condition holds [4], [5]. In a dissipative vector-Lyapunov formulation it has since been shown that (integral) input-to-state stability of a continuous-time comparison system carries over to the composite system [6], [7]. More generally, (vector-) comparison principles allow to infer stability properties from a usually lower order comparison system to a higher order nominal system.

In this note we show that the generalized small-gain condition for networks arising in the trajectory estimate and Lyapunov implication formulation of ISS is in fact necessary and sufficient for the input-to-state stability of an associated discrete-time comparison system. This result stands out from standard comparison principles, as it relates stability of an artificial discrete-time system with the stability of a nominal, continuous-time system, whereas classical comparison principles would compare continuous-time systems with continuous-time systems (or discrete-time systems with discrete-time systems). Furthermore, we may now utilize ISS Lyapunov functions to check generalized small-gain conditions, as is demonstrated in an example.

The next section defines necessary notation. Section III recalls two general small-gain theorems that give rise to a generalized, large-scale small-gain condition. In Section IV the comparison system is introduced, and the main result relating stability properties of this system to the generalized small-gain conditions is proved. An alternative approach to obtain parts of these main results is explained in Section V, where also a lemma on "dimension doubling" is provided, showing that it is no restriction to assume that gain matrices have a zero diagonal. An example demonstrating the use of an ISS Lyapunov function to verify the generalized small-gain condition and the conclusions follow in Sections VI and VII, respectively.

II. Notation

By $\mathbb{R}_+$ we denote the non-negative real numbers, and $\mathbb{R}_+^n$ is $(\mathbb{R}_+)^n$, the positive orthant in $\mathbb{R}^n$. This orthant induces a partial order on $\mathbb{R}^n$, which coincides with the component-wise order. For vectors $v, w \in \mathbb{R}_+^n$ we have $v \le w$ if for all i, $v_i \le w_i$. We have $v < w$ if $v \le w$ and $v \ne w$, and we write $v \ll w$ to denote that $v_i < w_i$ for all i.

By $w \oplus v$ we denote $\max\{w, v\}$ with respect to the component-wise ordering (i.e., a component-wise maximisation).

A continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$. The function Γ is of class $\mathcal{K}_\infty$, if in addition it is unbounded. By $\mathcal{KL}$ we denote the set of functions $\beta : \mathbb{R}_+^2 \to \mathbb{R}_+$ that are of class $\mathcal{K}$ in the first argument and decreasing in the second, such that for fixed $s \in \mathbb{R}_+$, $\lim_{r\to\infty} \beta(s, r) = 0$. By $\|\cdot\|_{L_\infty}$ we denote the essential supremum norm and $\|\cdot\|$ is the Euclidean norm.

III. Interconnected ISS systems

Consider $n \ge 2$ systems of the form

$$\dot{x}_i = f_i(x_i, u_1, \dots, u_n, w_i), \quad i = 1, \dots, n, \tag{1}$$

with $x_i, u_i \in \mathbb{R}^{N_i}, w_i \in \mathbb{R}^{M_i}$ and f_i satisfying the usual Carathéodory conditions for existence and uniqueness of solutions [8].

These individual systems are subject to the following coupling: For each i we let $u_j = x_j$ for $j \ne i$ and $u_i = 0$. This does not mean that each system necessarily depends on every other system (but it might). In fact, f_i may not depend on u_j. This will become more precise below.

Our aim is to provide a sufficient condition for ISS of the composite system, which is

$$\dot{x} = f(x, w) \tag{2}$$

with $x = (x_1^T, \dots, x_n^T)^T$, $w = (w_1^T, \dots, w_n^T)^T$, and

$$f(x, w) = (f_1(x_1, 0, x_2, \dots, x_n, w_1)^T, \dots, \\ f_n(x_n, x_1, \dots, x_{n-1}, 0, w_n)^T)^T.$$

We assume that each system is ISS and we give two qualitatively equivalent formulations how this can be stated: The trajectory formulation assumes that for each i there exist $\beta_i \in \mathcal{KL}$, $\gamma_{ij} \in (\mathcal{K}_\infty \cup \{0\})$, for $j \ne i$, and $\gamma_{iw} \in (\mathcal{K}_\infty \cup \{0\})$, such that for each initial condition (t_0, x_i^0) the solution of the ith system starting at (t_0, x_i^0) satisfies

$$\|x_i(t; t_0, x_i^0)\| \le \beta_i(\|x_i^0\|, t - t_0) + \\ \sum_{j \ne i} \gamma_{ij}(\|x_j\|_{L_\infty[t_0,t]}) + \gamma_{iw}(\|w_i\|_{L_\infty}) \,. \tag{3}$$

The Lyapunov implication formulation instead asks that for each i there exists a smooth function $V_i : \mathbb{R}^{N_i} \to [0, \infty)$ such that for some $\psi_i^1, \psi_i^2 \in \mathcal{K}_\infty$, $\psi_i^1(\|x_i\|) \le V_i(x_i) \le \psi_i^2(\|x_i\|)$ for all $x_i \in \mathbb{R}^{N_i}$, and that the following implication holds:

$$V_i(x_i) \ge \sum_{j \ne i} \gamma_{ij}(V_j(x_j)) + \gamma_{iw}(\|w_i\|) \\ \implies \langle \nabla V_i(x_i), f_i(x_i, x_1, \dots, x_n, w_i) \rangle < 0. \tag{4}$$

While these two formulations are qualitatively equivalent, the gains γ_{ij}, γ_{iw} do not need to be the same in (3),(4). Instead of (3) we could have taken the so-called "max-formulation"

$$\|x_i(t)\| \le \max_{j \ne i} \{\beta_i(\|x_i^0\|, t_0), \\ \gamma_{ij}(\|x_j\|_{L_\infty[0,t]}), \gamma_{iw}(\|w_i\|_{L_\infty})\} \,, \tag{5}$$

again with possibly different gains.

We see that the interconnection structure in our network is in fact described by the gains: Either there is a class $\mathcal{K}_\infty$ gain from one system to the other, indicating that there is an influence from one system to the other, or the respective gain is zero, indicating that there is no such influence. Thinking of the interconnection graph it is natural to write down a weighted adjacency matrix $\Gamma = (\gamma_{ij})_{i,j=1}^n$

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(thereby taking $\gamma_{ii} = 0$ for all i) to encode the interconnection structure in the network.

This so-called gain matrix induces a monotone map $\Gamma_\mu : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, which we simply call the gain operator, by

$$(\Gamma_\mu)_i(s) := \mu(\gamma_{i1}(s_1), \dots, \gamma_{in}(s_n))), \quad \text{for all } s \in \mathbb{R}^n_+ . \tag{6}$$

Here $\mu(s) = s_1 + \dots + s_n$ in case of (3),(4) and $\mu(s) = \max\{s_1, \dots, s_n\}$ in case of (5). In the following we will use the subscripts $+$ and $\oplus$ to make reference to one particular choice of μ. Building on essentially this setup the following theorems can be proved:

Theorem III.1 *Let interconnected systems* (1) *satisfying* (3) *or* (4) *be given. If there exists a $\mathcal{K}_\infty$ function α such that for the map $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $D(s)_i = s_i + \alpha_i(s_i)$, the condition*

$$(D \circ \Gamma_+)(s) \not\geq s, \quad \textit{holds for all } s \in \mathbb{R}^n_+, s \neq 0, \tag{7}$$

then system (2) *is ISS from w to x.*

Theorem III.2 *Let interconnected systems* (1) *satisfying* (5) *be given. If the condition*

$$\Gamma_\oplus(s) \not\geq s, \quad \textit{holds for all } s \in \mathbb{R}^n_+, s \neq 0, \tag{8}$$

then system (2) *is ISS from w to x.*

For proofs see [4],[5] or [9], and [10], [11], [12], respectively. For $a, b \in \mathbb{R}^n_+$ the condition $a \not\geq b$ means that the component-wise $\geq$ ordering should not hold, i.e., that for some i, $a_i < b_i$. The condition with $(D \circ \Gamma_\mu)(s) := D(\Gamma_\mu(s)) \not\geq s$ can be stated equivalently as $(\Gamma_\mu \circ D)(s) \not\geq s$, $s \neq 0$. For short we will sometimes write $M \not\geq \mathrm{id}$ if a monotone map $M : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ satisfies $M(x) \not\geq x$ for all $x \in \mathbb{R}^n_+, x \neq 0$.

IV. Comparison systems

It might seem odd to think of a comparison principle in anything else but a Lyapunov framework, but the connection is certainly there: Consider the systems

$$s^+ = \Gamma_+(s) + v\,, \tag{9}$$

and, respectively,

$$s^+ = \Gamma_\oplus(s) \oplus v\,. \tag{10}$$

Such a system is ISS if and only if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every $s(0) \in \mathbb{R}^n_+$, $\{v(k)\}_{k=1}^n \subset \mathbb{R}^n_+$, and every $k \geq 0$,

$$\|s(k)\| \leq \beta(\|s(0)\|, k) + \gamma(\sup_{l \geq 0} \|v(l)\|)\,. \tag{11}$$

We have the following main result, which immediately implies the discrete-time comparison principle for input-to-state stability of networks.

Theorem IV.1 *System* (9) *is ISS from v to s if and only if* (7) *holds. System* (10) *is ISS from v to s if and only if* (8) *holds.*

Corollary IV.2 (Comparison principle) *Let interconnected systems* (1) *satisfying* (3) *or* (4) *or, respectively,* (5) *be given. If the corresponding comparison system* (9) *or, respectively,* (10) *is ISS, then the composite system* (2) *is ISS from w to x.*

Note that in [4, Thm. 23] it already has been shown directly that if Γ is irreducible, then $\Gamma \not\geq \mathrm{id}$ and global asymptotic stability of the origin with respect to (9) for $v \equiv 0$ are equivalent. Our result extends this previous result to the ISS framework. For the difficult direction in the proof we show that systems (8),(9) are globally stable (GS) and have the asymptotic gain property (AG). Together, GS and AG are equivalent to ISS (see [13] for this result for continuous-time systems and [14], [15] for a discrete-time version).

We stress that the discrete time domain of the comparison systems (9),(10) has nothing to do with the time domain of the continuous-time nominal system (2).

Proof of Thm. IV.1: ISS implies the $\not\geq$ conditions: If systems (9),(10) are ISS, then in particular the origin is globally attractive with respect to autonomous dynamics. By a result in [16, Prop. 4.1] this implies that $\Gamma_\mu(s) \not\geq s$ for all $s \neq 0$.

Now we show that in case that (9) is ISS there exists an appropriate operator D such that (7) holds. By assumption there exists $\gamma \in \mathcal{K}_\infty$ such that for any $s(0) \in \mathbb{R}^n_+$ and input signal $v(\cdot) \leq v$ we have

$$\limsup_{k \to \infty} \|s(k)\|_1 \leq \gamma(\|v\|_1)\,, \tag{12}$$

where we have used the equivalence of norms on $\mathbb{R}^n$, $\|\cdot\|_1$ denoting the 1-norm. Define a function $\alpha \in \mathcal{K}_\infty$ by $\alpha(r) := \frac{1}{2n}\gamma^{-1}(r)$ and define $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by $D(s)_i = s_i + \alpha(s_i)$, $i = 1, \dots, n$, $s \in \mathbb{R}^n_+$.

Fix $r > 0$ and consider the set $S_r = \{s \in \mathbb{R}^n_+ : \|s\|_1 = \sum_i s_i = r\}$. By construction for $s \in S_r$ we have

$$D(s) = s + (\alpha(s_1), \dots, \alpha(s_n))^T \leq s + v_r\,, \tag{13}$$

with $v_r := (\frac{1}{2n}\gamma^{-1}(r), \dots, \frac{1}{2n}\gamma^{-1}(r))^T$. Observe that $\|v_r\|_1 = \frac{1}{2}\gamma^{-1}(r)$, or equivalently, $r = \gamma(2\|v_r\|_1) > \gamma(\|v_r\|_1)$. Now (12) implies that

$$(\Gamma_+)(s + v_r) \not\geq s\,, \quad \text{for all} \quad s \in S_r\,. \tag{14}$$

To see this assume the opposite. Then there exists an $s^* \in S_r$ such that $\Gamma_+(s^* + v_r) \geq s^*$. Consider the trajectory $\phi(\cdot)$ of the dynamical system $w^+ = \Gamma_+(w + v)$ with initial value $w(0) = s^*$ and input $v(k) \equiv v_r$. Assuming $w(k) \geq s^*$ we show inductively for $k \geq 0$ that $w(k+1) = \Gamma_+(w(k) + v(k)) \geq \Gamma_+(s^* + v_r) \geq s^*$. Since $\|s^*\|_1 = r > \gamma(\|v_r\|_1)$ we have a contradiction to (12), so (14) must hold.

Consider again an arbitrary $s \in S_r$. By (14) there exists an index $i \in \{1, \dots, n\}$ such that $s_i > (\Gamma_+(s + v_r))_i \geq (\Gamma_+(D(s)))_i$, the last inequality by (13). This implies

$$(\Gamma_+ \circ D)(s) \not\geq s\,, \quad \text{for all} \quad s \in S_r\,,$$

and since $r > 0$ was chosen arbitrarily and $\bigcup_{r>0} S_r = \mathbb{R}^n_+ \setminus \{0\}$ the claim follows.

The $\not\geq$-conditions imply ISS: We consider the summation formulation (7), respectively, (9) first. By [16, Thm. 5.10] condition (7) implies the existence of functions $\sigma_i \in \mathcal{K}_\infty$, $i = 1, \dots, n$, and $\tilde\alpha \in \mathcal{K}_\infty$ such that with $\sigma(r) = (\sigma_1(r), \dots, \sigma_n(r))^T$, $r \in [0, \infty)$, and $\tilde D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $\tilde D(s)_i = s_i + \tilde\alpha(s_i)$,

$$(\tilde D \circ \Gamma_+)(\sigma(r)) \ll \sigma(r)\,, \quad \text{for all } r > 0\,, \tag{15}$$

where $\ll$ denotes component-wise strictly less. Note that (15) can again be written equivalently as

$$(\Gamma_+ \circ \tilde D)(\sigma(r)) \ll \sigma(r)\,, \quad \text{for all } r > 0\,,$$

with the same $\tilde D$. Another ingredient is that by [4, Lemma 13] there exists a $\mathcal{K}_\infty$ function ϕ such that

$$(\mathrm{id} - \Gamma_+)(w) \leq v \quad \Longrightarrow \quad \|w\| \leq \phi(\|v\|)\,. \tag{16}$$

We start by showing that bounded inputs yield bounded trajectories: To this end assume that $v(k) \leq v \in \mathbb{R}^n_+$ for all $k \geq 0$. For any such v and arbitrary $s(0) \in \mathbb{R}^n_+$ by (15) there exists an $r > 0$ such that $\sigma(r) \geq s(0)$ and $\tilde A(\sigma(r)) \geq v$, where we denote $\tilde A = \tilde D - \mathrm{id}$.

Now assume that $s(k) \leq \sigma(r) + \tilde A(\sigma(r))$. This is obviously true for $k = 0$. For $k+1$ we compute

$$\begin{aligned} s(k+1) = \Gamma_+(s(k)) + v(k) &\leq \Gamma_+(\sigma(r) + \tilde A(\sigma(r))) + v \\ &\leq \Gamma_+(\sigma(r) + \tilde A(\sigma(r))) + \tilde A(\sigma(r)) \\ &\leq \sigma(r) + \tilde A(\sigma(r))\,. \end{aligned}$$

So by induction it follows that the trajectory $s(\cdot)$ is bounded.

Now for fixed initial condition $s(0)$ and input bounded by $v(\cdot) \le v$ let

$$\begin{aligned} s^* := s^*(s(0), v) &:= \sup_{k\ge 0} s(k) \\ &\le \sup_{k\ge 0}\{s(0), \Gamma_+(s(k)) + v(k)\} \\ &\le \max\{s(0), \Gamma_+(s^*) + v\} \le s(0) + \Gamma_+(s^*) + v\,. \end{aligned}$$

In other words, $(\mathrm{id} - \Gamma_+)(s^*) \le s(0) + v$. So by (16) we have $\|s^*\| \le \phi(\|s(0)+v\|) \le \phi(\|s(0)\|+\|v\|) \le (\phi\circ(\mathrm{id}+\rho))(\|s(0)\|) + (\phi\circ(\mathrm{id}+\rho^{-1}))(\|v\|)$, the last step by a variant of the weak triangle inequality [2], [10] with an arbitrary $\rho \in \mathcal{K}_\infty$. In the last estimate the functions $\phi\circ(\mathrm{id}+\rho)$ and $\phi\circ(\mathrm{id}+\rho^{-1})$ are of class $\mathcal{K}_\infty$, so we have obtained the GS property.

For the asymptotic behavior of a trajectory we obtain

$$\begin{aligned} s^\# &:= \limsup_{k\to\infty} s(k) = \limsup_{k\to\infty} \big(\Gamma_+(s(k)) + v(k)\big) \\ &\le \Gamma_+(s^\#) + v\,. \end{aligned}$$

It follows that $\|s^\#\| \le \phi(\|v\|)$, and this is the AG property, which together with the GS property is equivalent to ISS (cf. [14, Thm. 2] or [15, Thm. 1], where AG is named $\mathcal{K}$-asymptotic gain and GS is named UBIBS).

The proof that (8) implies ISS of system (10) is slightly different, but similar. The necessary ingredients are given by [16, Thm. 6.4], which says that (8) is equivalent to the following two statements:

1) There exist functions $\sigma_i \in \mathcal{K}_\infty$ such that (with the previous notation),
$$\Gamma_\oplus(\sigma(r)) \ll \sigma(r)\,, \quad \text{for all } r > 0\,.$$
2) There exists a function $\phi \in \mathcal{K}_\infty$ such that
$$w \le \Gamma_\oplus(w) \oplus v \quad \Longrightarrow \quad \|w\| \le \phi(\|v\|)\,.$$

The remainder of the proof is essentially a repetition of the arguments given above. ■

V. Related approaches

One reviewer was kind enough to point out that the "max"-version contained in Theorem IV.1 and Corollary IV.2 can be derived using an alternative approach based on the results presented in [17], [18]. The argument is as follows.

Let $\gamma_{ij}, \gamma_{iw} \in \mathcal{K}_\infty \cup \{0\}$, $i,j = 1,\dots,n$ be given and consider the continuous-time, time-delay system given by

$$x_i(t) = \max_j \gamma_{ij}(x_i(t-1)) \oplus u_i(t), \quad i = 1,\dots,n, \tag{17}$$

with $x_i(\cdot), u_i(\cdot) \ge 0$, $i = 1,\dots,n$. Notably, (17) is a functional difference equation. It's relation to system (10) is as follows: For any trajectory $\{s(k)\}_{k\ge 0}$ of system (10), the initial conditions $x(t) := (x_1(t),\dots,x_n(t))^T = s(0)$ for all $t \in (-1,0]$ and the inputs defined by $u(t) := (u_1(t),\dots,u_n(t))^T = v(k)$ for all $t \in (k-1,k]$ and integers $k \ge 0$, produce a trajectory $x(\cdot)$ that satisfies at integer times

$$x(k) = s(k) \text{ for all } k \ge 0. \tag{18}$$

Now, Theorem 3.1 in either of [17], [18] asserts that condition (8) (equivalently, $\Gamma_\oplus(s) \ge s \implies s = 0$) implies the ISS property of the interconnected system (17) and hence of system (10) by virtue of the correspondence (18). Moreover, [17], [18] provide a formula for the ISS gain of system (10). This result extends to the fact that any discrete-time system

$$x^+ = F(x,u), \quad F: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,$$

is ISS, for which an estimate of the form

$$\lfloor F(x,u) \rfloor \le \Gamma_\oplus(\lfloor x \rfloor) \oplus G(\lfloor u \rfloor)$$

holds, where $\Gamma_\oplus \not\ge \mathrm{id}$ is as above, $G: \mathbb{R}^m_+ \to \mathbb{R}^n_+$ is monotone and satisfies $G(0) = 0$, and $\lfloor x \rfloor$ denotes the component-wise absolute value of $x \in \mathbb{R}^n$, i.e., $\lfloor x \rfloor = \max\{x, -x\}$.

Two more remarks are in order. The first is an obvious fact, but for sake of completeness, we state it as a remark.

Remark V.1 *If the discrete time system*

$$x^+ = F(x) \oplus v \tag{19}$$

is ISS from v to x, where $F: \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is monotone and continuous, then for any monotone and continuous map $G: \mathbb{R}^n_+ \to \mathbb{R}^n_+$ such that $G(0) = 0$, the system

$$x^+ = F(x) \oplus G(v) \tag{20}$$

is also ISS from v to x. Just the gain will be different and in general depend on G. An analogous statement is true when $\oplus$ is replaced by $+$ in (19),(20).

The second remark regards our general assumption that for $\Gamma = (\gamma_{ij})_{i,j=1}^n$ we have $\gamma_{ii} = 0$ for $i = 1,\dots,n$.

Remark V.2 *The works [17], [18] explicitly consider the case $\gamma_{ii} \ne 0$. For the purpose of the results in these references, this case is indeed meaningful. In fact, all results in the literature on operators of the form $\Gamma_+, \Gamma_\oplus: \mathbb{R}^n_+ \to \mathbb{R}^n_+$ induced by matrices of $\mathcal{K}$ functions, allow for this extension with no or minimal modifications, including those results in [4], [5], [10], [11], [12], [16]. An alternative to these modifications is the following Lemma V.3. However, for the results in this note, which provide re-interpretation of Theorems III.1 and III.2 in terms of comparison principles, the case $\gamma_{ii} \ne 0$ is not interesting, because it is explicitly excluded by those theorems. Nevertheless, for the sake of completeness we note that Theorem IV.1 also holds in the case that the functions γ_{ii} are chosen to be of class $\mathcal{K}_\infty$.*

The following lemma shows that for the purpose of the results presented here or in [4], [5], [10], [11], [12], [16], [17], [18], one can, without loss of generality, always assume that the diagonal of a gain matrix $\Gamma = (\gamma_{ij})_{i,j=1}^n$ is zero. It should be noted that a similar idea has been used before in [4, Remark 5.8].

Lemma V.3 (Dimension doubling) *Let $D, T: \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone maps. Define $\overline{D}, S: \mathbb{R}^{2n}_+ \to \mathbb{R}^{2n}_+$ via*

$$\overline{D}\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} D(x) \\ D(y) \end{pmatrix} \quad \text{and} \quad S\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} T(y) \\ T(x) \end{pmatrix}\,. \tag{21}$$

Then $D \circ T \not\ge$ id if and only if $\overline{D} \circ S \not\ge$ id.

Note that in particular the case $D = \mathrm{id}$ is included, and we do not require continuity. The map S can be thought of as the matrix $\begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}$ and as such clearly has a zero-diagonal. A particular consequence is that all results based on the condition $T \not\ge \mathrm{id}$, $(D\circ T) \not\ge \mathrm{id}$, or one of their equivalent formulations (like the cycle condition in one case, cf. [16]), are true also for operators induced by matrices of gains that have non-zero diagonal. Via the construction (21) in Lemma V.3, such matrices with non-zero diagonal can always be embedded into matrices with zero diagonal, satisfying the same type of stability condition. This extends in particular the results in [4], [5], [10], [16], which throughout assumed that diagonal entries of gain matrices be zero.

Proof: We will use the fact that by [16, Lemma 2.1] $M \not\ge$ id implies that for all positive integers k also $M^k \not\ge$ id. Here M^k denotes k-times application of M.

For the first part of the proof, assume that $D \circ T \not\geq \mathrm{id}$, but that there exists a $z = (x^T, y^T)^T \in \mathbb{R}^{2n}_+, z \neq 0$, so that $(\overline{D} \circ S)(z) \geq z$. By monotonicity then also $(\overline{D} \circ S)^2(z) \geq z$, implying that $(D \circ T)^2(x) \geq x$ and $(D \circ T)^2(y) \geq y$. Since at least one of x and y is non-zero, this contradicts the assumption, so we must have $\overline{D} \circ S \not\geq \mathrm{id}$.

The second part is similar: Assume that $\overline{D} \circ S \not\geq \mathrm{id}$ but that there exists an $0 \neq x \in \mathbb{R}^n_+$ such that $(D \circ T)(x) \geq x$. Let $z = (x^T, x^T)^T$, then $(\overline{D} \circ S)(z) = \begin{pmatrix} (D \circ T)(x) \\ (D \circ T)(x) \end{pmatrix} \geq \begin{pmatrix} x \\ x \end{pmatrix} = z$. The existence of this z contradicts the assumption, so we must have $D \circ T \not\geq \mathrm{id}$. ∎

VI. Example

In this section we demonstrate the usefulness of Theorem IV.1, by considering a nontrivial $\Gamma_+ : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ and asking, whether or not there exists a $\rho \in \mathcal{K}_\infty$, such that for $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by $D(s)_i = s_i + \rho(s_i)$ we have $D \circ \Gamma_+ \not\geq \mathrm{id}$. To give a positive answer, it is now sufficient to find an ISS Lyapunov function for system (9).

Example VI.1 *Let $n \geq 2$. Define $\Gamma_+ : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by*

$$\big(\Gamma_+(s)\big)_i = \frac{1}{4}\big(s_{i-1}^{1/i} + s_{i+1}^{i+1}\big),$$

with the convention that $s_0 = s_{n+1} = 0$. So, in the case $n = 5$ we have

$$\Gamma_+(s) = \frac{1}{4}\begin{pmatrix} s_2^2 \\ \sqrt{s_1} + s_3^3 \\ \sqrt[3]{s_2} + s_4^4 \\ \sqrt[4]{s_3} + s_5^5 \\ \sqrt[5]{s_4} \end{pmatrix}.$$

Claim: The function $V(s) := \max_i s^{i!}$ is an ISS Lyapunov function for (9)*, i.e., there exist functions $\psi_1, \psi_2, \alpha, \gamma \in \mathcal{K}_\infty$ such that*

1) *$\psi_1(\|s\|) \leq V(s) \leq \psi_2(\|s\|)$, and*
2) *$V(\Gamma_+(s) + v) - V(s) \leq -\alpha(\|s\|) + \gamma(\|v\|)$ for all $s \geq 0$ and $u \geq 0$.*

It is well known (see, e.g., [15, Thm. 1]) that a discrete time system is ISS if and only if it admits a smooth ISS Lyapunov function. Moreover, the existence of a merely continuous ISS Lyapunov function is sufficient for ISS [15, Lemma 3.5].

Proof of the claim: The existence of $\psi_1, \psi_2 \in \mathcal{K}_\infty$ is obvious. For the first estimate in the second part we use the inequality $(a + b + c)^k \leq 3^k \max\{a^k, b^k, c^k\}$, which holds for all real $a, b, c \geq 0$ and all positive integers k. Maximization over i is always understood over the range $\{1, \ldots, n\}$.

Now consider

$$V(\Gamma_+(s) + v) - V(s) = \max_i \left(\frac{s_{i-1}^{1/i}}{4} + \frac{s_{i+1}^{i+1}}{4} + v_i\right)^{i!} - \max_i s_i^{i!}$$
$$\leq \max_i \max\left\{\left(\frac{3}{4}\right)^{i!} s_{i-1}^{(i-1)!}, \left(\frac{3}{4}\right)^{i!} s_{i+1}^{(i+1)!}, (3v_i)^{i!}\right\} - \max_i s_i^{i!}$$
$$= \max_i \max\left\{\left(\frac{3}{4}\right)^{(i+1)!} s_i^{i!}, \left(\frac{3}{4}\right)^{(i-1)!} s_i^{i!}, (3v_i)^{i!}\right\} - \max_i s_i^{i!}$$
$$\leq \max_i \max\left\{\left(\frac{3}{4}\right)^{(i-1)!} s_i^{i!}, (3v_i)^{i!}\right\} - \max_i s_i^{i!}$$
$$\leq \frac{3}{4}\max_i s_i^{i!} - \max_i s_i^{i!} + \max_i (3v_i)^{i!} \leq -\frac{1}{4}\max_i s_i^{i!} + \max_i (3v_i)^{i!}$$
$$\leq -\frac{1}{4}\min\big\{\max_i s_i, \max_i s_i^{n!}\big\} + 3^{n!}\max\{\max_i v_i, (\max_i v_i)^{n!}\}$$
$$= -\frac{1}{4}\min\big\{\|s\|_\infty, \|s\|_\infty^{n!}\big\} + 3^{n!}\max\big\{\|v\|_\infty, \|v\|_\infty^{n!}\big\}.$$

It follows that V is indeed an ISS Lyapunov function. By application of Corollary IV.2 it can now be deduced that any large-scale interconnection (2) *of ISS subsystems* (1) *with a gain operator given by Γ_+ is ISS with respect to external inputs.*

VII. Conclusions

In this note we have shown that generalized small-gain conditions and comparison principles go hand in hand for ISS systems. Both are analysis tools providing sufficient conditions for the stability of coupled or large-scale systems. By showing that these conditions are equivalent for the ISS case we have bridged the gap between these seemingly different analysis tools. By means of an example we have shown that discrete-time ISS Lyapunov functions are now at our disposal to check generalized small-gain conditions as they have been stated in [4], [5]. Extensions to more general stability frameworks like input-to-output stability (e.g., [19]) are straight-forward.

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