Discussion of “On a small gain theorem for ISS networks in dissipative Lyapunov form”

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In the paper by Dashkovskiy, Ito, and Wirth the construction of Lyapunov functions for nonlinear composite systems is pursued. These systems consist of usually more than two interconnected subsystems for which input-to-state stability (ISS) Lyapunov functions are assumed to be known. Yet, so far, the stability of the overall system, or equivalently, the existence of a Lyapunov function for the composite system is not guaranteed. Instead, a sufficiency condition which generalizes the well known small-gain condition

$$\gamma_{12} \circ \gamma_{21} < \mathrm{id} \tag{1}$$

on the gains between only to subsystems is required to allow for the geometrical construction of the desired overall Lyapunov function. Roughly speaking, the generalized small-gain condition requires that all *cycle gains* have to satisfy a contraction property like (1) in a robust sense, as was recently illuminated in [3].

The aim of this discussion is to put the spotlight on some related open problems which open up new research directions.

The elegance of the result by Dashkovskiy, Ito, and Wirth lies in the geometrical construction of the overall Lyapunov function: Previous contributions have shown stability directly without the use of a Lyapunov function [1, 6, 9, 12] or provided constructions of an only locally Lipschitz continuous Lyapunov function [2] (but see also [10] for more details on the geometrical aspects of the construction), with the option of a separate smoothing step [7]. In this work, one of the proposed constructions does provide an inherently smooth Lyapunov function from smooth Lyapunov functions for subsystems and using an integration technique previously successfully used by the second author for constructions involving interconnections of two subsystems [5].

It may be expected that the sufficiency condition permitting such a smooth construction should be the same as in the nonsmooth case, as it is known that in the presence of (integral) input-to-state stability a smooth Lyapunov function must exist. This conjecture is also nurtured by the geometrical consideration, that the scaling functions underlying the smooth and the nonsmooth constructions can be regarded as nonlinear versions of left and, respectively, right eigenvectors, and that the stability condition only regards the spectrum of an associated operator.

However, to date the question whether the generalized small-gain conditions in the paper are in fact equivalent and whether the associated left and right eigenvector existence problems are equivalent, remains unanswered.

An interesting side remark regards the various literature on extensions of spectral theory to nonlinear operators: While there have been many publications considering eigenvalues and right eigenvectors of various types of nonlinear mappings (for homogeneous mappings see e.g. [4] and for concave mappings see [8]), there does not seem to exist a corresponding literature on the problem of finding left eigenvectors for nonlinear mappings.

Related to these geometrical constructions of Lyapunov functions are also the dynamics of the lower dimensional comparison systems aris-

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ing from ISS Lyapunov stability estimates of the subsystems. Using the notation of the paper we can write down a differential equation

$$\dot{v} = -A(v) + \Gamma(v), \quad v \in \mathbb{R}^n_+, \tag{2}$$

relating the stability estimates of the subsystems. The order of this system is n — the number of interconnected subsystems. It can be shown that stability properties of this comparison system can be translated to the same stability properties for the original composite system, as undertaken in [11]. In a sense, cooperative systems like (2) are at the core of large-scale systems analysis in the ISS framework, and it can be expected that the constructions in this paper can be translated back to comparison systems.

The geometric construction in the present paper further motivates the stability analysis and tailored design of more general networks of nonlinear systems with corresponding non-cooperative comparison systems. While input-to-state stability and related concepts provide worst-case estimates, the resulting theory will always be conservative for systems where the nature of feedback is not that of a disturbance but can have beneficial effects on the stability of the system. Especially in contexts where synchronization or consensus is desired, interconnection topologies cease to yield purely cooperative comparison dynamics and it remains to future research to extend constructive methods as detailed in this paper to this broader scope.

References

[1] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS small-gain theorem for general networks. *Math. Control Signals Syst.*, 19(2):93–122, May 2007.

[2] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM J. Control Optim.*, 48(6):4089–4118, 2010.

[3] Sergey N. Dashkovskiy, Michael Kosmykov, and Fabian R. Wirth. A small gain condition for interconnections of ISS systems with mixed ISS characterizations. *IEEE Trans. Autom. Control*, 2011. To appear, DOI: 10.1109/TAC.2010.2080110.

[4] Stéphane Gaubert and Jeremy Gunawardena. The Perron-Frobenius theorem for homogeneous, monotone functions. *Trans. Amer. Math. Soc.*, 356(12):4931–4950 (electronic), 2004.

[5] Hiroshi Ito. State-dependent scaling problems and stability of interconnected iISS and ISS systems. *IEEE Trans. Autom. Control*, 51(10):1626–1643, 2006.

[6] Z.-P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Math. Control Signals Systems*, 7(2):95–120, 1994.

[7] Zhong-Ping Jiang, Iven M. Y. Mareels, and Yuan Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica J. IFAC*, 32(8):1211–1215, 1996.

[8] Ulrich Krause. Concave Perron-Frobenius theory and applications. *Nonlinear Anal.*, 47(3):1457–1466, 2001.

[9] Henry G. Potrykus, Frank Allgöwer, and S. Joe Qin. The character of an idempotent-analytic nonlinear small gain theorem. In *Positive systems (Rome, 2003)*, volume 294 of *Lecture Notes in Control and Inform. Sci.*, pages 361–368. Springer, Berlin, 2003.

[10] B. S. Rüffer. Monotone inequalities, dynamical systems, and paths in the positive orthant of Euclidean n-space. *Positivity*, 14(2):257–283, June 2010.

[11] B. S. Rüffer, C. M. Kellett, and S. R. Weller. Connection between cooperative positive systems and integral input-to-state stability of large-scale systems. *Automatica J. IFAC*, 46(6):1019–1027, 2010.

[12] Andrew R. Teel. Input-to-state stability and the nonlinear small gain theorem. Private communication, 2005, 2005.