Computational comparison principles for large-scale system stability analysis*

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Abstract: Stability analysis of complex and large-scale systems is often aided by some form of model reduction, ideally down to a one-dimensional system via a Lyapunov function. In this context comparison principles arise very naturally. If the comparison system can be shown to be monotone, then an extension of a homotopical fixed point algorithm can be used to verify practical quasi-global asymptotic stability of the composite nominal system. This method is applied to a class of nonlinear examples.

Keywords: Large-scale systems, nonlinear systems, Lyapunov functions, comparison principle.

1 Introduction

Stability analysis and decentralized control of large-scale or compartmental systems has a long history, see e.g. [16] for a 40 year old survey article on the subject or the textbooks [5, 8, 17]. Recent advances in nonlinear stability theory with concepts like input-to-state stability (ISS) [19] and its various relatives have called for an extension of classical large-scale results to these more general frameworks, where systems participating in interconnections may satisfy only weak stability properties like integral-input-to-state stability (iISS), see e.g. [4].

The past two decades have also seen the emergence of numerical methods that can aid in the stability analysis of interconnected systems, most notably efficient solvers for linear matrix inequalities (LMI), that together with integral quadratic constraints (IQC) [2, 7] and sum of squares (SOS) relaxation [11] provide powerful tools to aid this type of analysis.

An alternative, analytical approach is that of comparison principles, vector Lyapunov functions, and generalized small-gain conditions [1, 4, 5, 8, 9, 10, 13, 14, 15, 17]. Motivated by a lack of computational tools associated with this approach for inherently nonlinear problems, here we explain how effective algorithms can be used to verify stability properties of large-scale systems via comparison principles. This provides a large-scale stability criterion which is not only of a theoretical nature, but also particularly suited for applications: First of all, it deals with practical and quasi-global stability, which is of greater generality than "pure" global stability concepts and hence of wider applicability in applications. Secondly, the approach provides a numerical scheme to check the stability condition, a promising development that aligns applicability of comparison principles with IQC and LMI approaches.

In this paper we first recall some facts about monotone systems that will be subsequently utilized for comparison systems. Then we define the class of large-scale systems in Section 3 together with corresponding comparison systems. Sections 4 and 5 introduce the quasi-global practical stability concepts, state a stability criterion, and then formulate an algorithm that can be used for numerical stability verification. A numerical example is briefly discussed in Section 6.

2 Preliminaries

Here we provide a rather informal recollection of a number of facts about monotone systems from a range of references [5, 15, 18], along with appropriate notation used throughout. The set $\mathbb{R}_+$ denotes the nonnegative real numbers, so $\mathbb{R}^n_+$ is the *positive orthant* in $\mathbb{R}^n$, and it defines the following component-wise partial order: Vectors $v, y \in \mathbb{R}^n$ are ordered by $v \leq y$ if $y - v \in \mathbb{R}^n_+$, $v \ll y$ if $y - v \in \operatorname{int} \mathbb{R}^n_+$, where $\operatorname{int} A$ denotes the interior of a set A, and $v < y$ if $v \leq y$ and $v \neq y$.

Consider an autonomous, continuous-time, and possibly nonlinear dynamical system that evolves on $\mathbb{R}^n_+$,

$$\dot{v} = g(v). \tag{1}$$

Assume that g is locally Lipschitz, so that solutions exist and are unique.

Assumption 2.1 *The function g is* quasi-monotone nondecreasing*, which is the same as* "type K," *i.e., for all $v, y \in \mathbb{R}^n_+$ and all $i \in \{1, \dots, n\}$, $g_i(v) \leq g_i(y)$, whenever $v \leq y$ and $v_i = y_i$.*

Assumption 2.2 *The origin is attractive with respect to* (1) *with basin of attraction $\mathcal{B}$.*

Under Assumptions 2.1 and 2.2 the following implications hold:

1. It holds that $g(v) \not\geq 0$ for all $v \in \mathcal{B}, v \neq 0$.
2. The set $\Omega := \{v \in \mathbb{R}^n_+ \colon g(v) \ll 0\}$ satisfies that for all $r > 0$ sufficiently small, $\Omega \cap S_r \neq \emptyset$, where $S_r := \{v \in \mathbb{R}^n_+ \colon \|v\|_1 = \sum_i v_i = r\}$.
3. The origin is stable in the sense of Lyapunov.
4. If in Assumption 2.2 $\mathcal{B} = \mathbb{R}^n_+$, i.e., the origin is globally asymptotically stable, then for all $r > 0$, $\Omega \cap S_r \neq \emptyset$.

*This research was supported under Australian Research Council's *Discovery Projects* funding scheme (project number DP0880494) as well as by the Japan Society for the Promotion of Science (JSPS).

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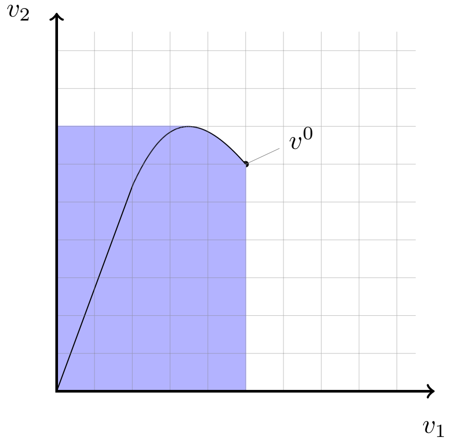

Figure 1: A trajectory of the system $\dot{v} = g(v)$ on $\mathbb{R}^2_+$ with $g(v) = (-v_1 + v_2/4, -v_2 + 2v_1)^T$, starting in $v^0 = (2.5, 3)^T$. Since g is type K and $\phi(t, v^0) \longrightarrow 0$ as $t \longrightarrow \infty$, we deduce that in the shaded region it must hold that $g(v) \ngeq 0$.

Further properties of system (1) are as follows: Due to the type K condition we have the following *ordering of solutions*: Let $v^0, v^1 \in \mathbb{R}^n_+$, then on the maximal interval $J = [0, T)$ where both solutions $\phi(\cdot, v^0)$ and $\phi(\cdot, v^1)$ of (1) exist, the following implications hold for $t \in J$,

1. if $v^0 \leq v^1$ then $\phi(t, v^0) \leq \phi(t, v^1)$;
2. if $v^0 < v^1$ then $\phi(t, v^0) < \phi(t, v^1)$; and
3. if $v^0 \ll v^1$ then $\phi(t, v^0) \ll \phi(t, v^1)$.

As a consequence it is possible to show that if $g \ngeq 0$ for all $v \in \mathbb{R}^n_+, v \neq 0$, the set Ω is positively invariant, i.e., $v^0 \in \Omega$, then $\phi(t, v^0) \in \Omega$ for all $t \geq 0$.

Also it follows that, as illustrated by Figure 2, if g is type K and $\phi(t, v^0) \longrightarrow 0$ for some $v^0 \in \mathbb{R}^n_+$ then the region $B(v^0) := \{v \in \mathbb{R}^n_+ : v \leq \phi(t, v^0) \text{ for some } t \geq 0\}$ is contained in $\mathcal{B}$, the region of attraction of the origin. Hence, integrating one trajectory can be enough to obtain the following conclusion.

Lemma 2.3 *Let g be of type K and locally Lipschitz with $g(0) = 0$. If for some $v^0 \in \mathbb{R}^n_+$, $\phi(t, v^0) \longrightarrow 0$, then the origin is asymptotically stable (AS) with respect to* (1) *and $B(v^0) \subset \mathcal{B}$.*

A more topological reasoning underlies the following, global result.

Lemma 2.4 *Let g be of type K and locally Lipschitz with $g(0) = 0$. If Ω is unbounded in every coordinate direction, i.e., that for any $v \in \mathbb{R}^n_+$ one can find a $y \in \Omega$, so that $y \geq v$, then the origin must be globally asymptotically stable (GAS) with respect to* (1).

The only difficulty in strengthening AS to GAS is to verify that Ω satisfies this unboundedness property. At this point it should be noted that there exist examples of functions g satisfying all the above assumptions, except that Ω is not unbounded in *all* coordinate directions. So the reasoning proposed here, is not applicable for any monotone system. However, as we will see later, it is applicable for quite a wide range of monotone systems.

3 Large-scale and comparison systems

Consider a large-scale system

$$\dot{x} = f(x) \tag{2}$$

with f locally Lipschitz and decomposed into subsystems $\dot{x}_i = f_i(x)$, $x_i \in \mathbb{R}^{N_i}$, $i = 1, \ldots, n$, with $x = (x_1^T, \ldots, x_n^T)^T$. A classical approach to prove stability properties of such a large-scale system is by means of vector Lyapunov functions and comparison principles.

Assume we already have found Lyapunov functions $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ satisfying for some $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, $\psi_{i1}(\|x_i\|) \leq V_i(x_i) \leq \psi_{i1}(\|x_i\|)$ as well as the dissipation inequalities

$$\langle \nabla V_i(x_i), f_i(x) \rangle \leq g_i(V_1(x_1), \ldots, V_n(x_n)). \tag{3}$$

Assume $g = (g_1, \ldots, g_n)^T : \mathbb{R}^n_+ \to \mathbb{R}^n$ is locally Lipschitz and of type K. If g is differentiable, then the type K requirement can be checked easily: The Jacobian $Jg(v)$ has to have non-negative entries off the diagonal. In the locally Lipschitz case, g is differentiable almost everywhere, and it is enough that the Jacobian matrix $Jg(v)$ has non-negative off-diagonal entries almost everywhere for g to be type K. For later use we introduce the shorthand notation $\underline{V}(x) := (V_1(x_1), \ldots, V_n(x_n))^T$.

4 Practical quasi-global asymptotic stability

We are interested in numerical methods to check stability properties of large-scale systems. For this reason we resort to quasi-global and practical stability concepts: Our methods will not be able to produce inherently global results, we will only be able to assert that a arbitrary large set belongs to the region of attraction.

Similarly, we cannot guarantee that the origin is in fact asymptotically stable, due to numerical precision issues. Instead, we can only assert asymptotic stability of a very small compact set containing the origin. Hence the name practical stability.

Definition 4.1 *Consider a system $\dot{x} = f(x)$. The origin is termed* practically quasi-global asymptotically stable *(PQGAS) if there exists a large set $\mathcal{R}$ and a small compact set $\mathcal{C} \subset \mathcal{R}$ such that*

1. *$\mathcal{R}$ is forward-invariant under* (1)*;*
2. *for any $x^0 \in \mathcal{R}$ the trajectory $x_{(1)}(t; x^0)$ eventually enters $\mathcal{C}$, i.e., that $\lim_{t\to\infty} \inf_{c\in\mathcal{C}} \|x_{(1)}(t; x^0) - c\| = 0$.*

In practice, we will try to find a large set $\mathcal{R}$. Our aim is to infer this stability property from the comparison system (1) to the large-scale system (2) given together with (3), which is achieved by the following comparison principle.

Theorem 4.2 (Comparison principle) *Let f and Lyapunov functions V_i be given as in Section 3. Assume the dissipative inequalities* (3) *hold with g of type K and locally Lipschitz.*

Assume the origin is practically quasi-global asymptotically stable with respect to (1) *on a region $\mathcal{R}_{(1)} \subset \mathbb{R}^n_+$ with $\mathcal{C}_{(1)} \subset \mathcal{R}_{(1)}$. Then the origin is also practically quasi-global asymptotically stable with respect to* (2) *on the region $\mathcal{R}_{(2)} = \underline{V}^{-1}[\mathcal{R}_{(1)}] \subset \mathbb{R}^N$ with $\mathcal{C}_{(2)} = \underline{V}^{-1}[\mathcal{C}_{(1)}] \subset \mathcal{R}_{(2)}$.*

The proof follows well-known comparison principle type results, and is omitted for brevity. The main argument is based on monotonicity of the solutions of the comparison system $\dot{v} = g(v)$, where the dynamics of v serves as an estimate of the behaviour of the vector $\underline{V}(x(t))$, as t evolves.

To be able to actually use Theorem 4.2, we have to specify the regions $\mathcal{R}$ and $\mathcal{C}$ for the comparison system. Since this system is monotone, we may utilize a homotopical fixed point algorithm [3] implementing the famous KKM Lemma [6] for this task, which amounts to property 2) in Section 2. The next section describes how this can be formalized and implemented on a computer.

5 Algorithmic framework

Combining Lemma 2.3 with Theorem 4.2 leads us to a numerical algorithm to verify stability properties of a large-scale system (2) as follows. If the origin is attractive for a monotone system (1), then necessarily $g(v) \not\geq v$ for all $v > 0$. In particular, this holds on any set S_r, where $r > 0$. Using an algorithm due to Eaves [3], for any given $r > 0$, $g : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ satisfying $g(v) \not\geq v$ for all $v \in S_r$, this algorithm computes a $v_{\mathrm{KKM}} \in S_r$ satisfying $g(v_{\mathrm{KKM}}) \ll 0$. It will produce the point v_{KKM} if the prerequisites are met, and if it does not produce such a point, then this requirement might not be met.

Given the point v_{KKM}, our candidate region $\mathcal{R}$ is the order interval $[0, v_{\mathrm{KKM}}]$. From here we proceed with the following algorithm, which essentially integrates the system forward to find $\mathcal{C}$.

A few remarks regarding the algorithm: The convergence in step 4 is understood as up to numerical precision. The numerical error analysis resulting from the particular integration method has to be taken into account for the assignments of the regions $\mathcal{R}$ and $\mathcal{C}$. The computed trajectory of (1) starting in v_{KKM} is denoted by $v_t, t \geq 0$. The right hand sides of the assignments in steps 6,7 are order intervals (sets). The lines stating *Ensure* are sanity checks. The conditions states in these lines should be satisfied at these points in the algorithm. If not, this might be either due to fact that we did not start with a KKM-point to begin with, that the origin (or a small compact set $\mathcal{C}$ is not attractive, or that the precision of our numerical integration is not high enough. If the algorithm succeeds, then the origin is practically quasi-global asymptotically stable with respect to the system (1), with the sets $\mathcal{C}$ and $\mathcal{R}$ given by the algorithm. Notably, we can try this algorithm for arbitrarily large $r > 0$, hence the naming quasi-global, is justified at the end.

Algorithm 1 Verify PQGAS of a monotone system (1)

Require: radius $r > 0$ large
Require: g is of type K, locally Lipschitz
1: $v_{\mathrm{KKM}} \leftarrow$ Eaves-KKM-Algorithm(S_r)
Ensure: $v_{\mathrm{KKM}} \gg 0$ and $g(v_{\mathrm{KKM}}) \ll 0$ and $g(0) = 0$
2: $t \leftarrow 0$
3: $v_t \leftarrow v_{\mathrm{KKM}}$
4: Integrate $\dot{v} = g(v)$ forward until v_t converges
Ensure: that v_t decreases monotonically in all components
5: $v_{\mathrm{final}} \leftarrow v_t$
Ensure: $0 \leq v_{\mathrm{final}} \leq v_{\mathrm{KKM}}$
6: $\mathcal{R} \leftarrow [0, v_{\mathrm{KKM}}]$
7: $\mathcal{C} \leftarrow [0, v_{\mathrm{final}}]$

Remark 5.1 *It should be noted, that as initial condition any other point instead of v_{KKM} could be used. We mainly resort to this choice of initial condition, because it guarantees existence of the corresponding solution for all times. If in the algorithm we would start integrating in some v^0 that does not necessarily satisfy $g(v^0) \ll 0$, then extra care has to be taken to ensure that finite escape times will be detected by numerical integration scheme. Also, there might be strong transient behaviour, that first takes the trajectory $\phi(\cdot, v^0)$ far away from the origin, before it might finally approach the origin. This would provide a larger region $\mathcal{R}$, cf. Fig. 2. However, depending on the type of nonlinearities and together with unavoidable numerical inaccuracies, this might lead to wrong conclusions.*

6 Example

Here we are going to show some numerical examples for some randomly chosen type K function g satisfying the $\not\geq$ condition. The resulting system can be interpreted as the comparison system to a large-scale system (2) consisting of n subsystems. To this end assume that $A \in \mathbb{R}^{n \times n}$ is of the form $A = -I + P$, where I is the identity and P is a nonnegative matrix (element-wise), with spectral radius $\rho(P) := \max\{|\lambda| \colon \lambda \text{ is an eigenvalue of } P\} < 1$. It can be verified that the spectral abscissa $\alpha(A)$ satisfies

$$\begin{aligned}\alpha(A) &:= \max\{\operatorname{Re} \lambda \colon \lambda \text{ is an eigenvalue of } A\} \\ &= -1 + \rho(P) < 0.\end{aligned}$$

So A is a Hurwitz matrix with negative diagonal entries and non-negative off-diagonal entries. Now we define a nonlinear but smooth and order preserving coordinate transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $S(0) = 0$ and $S(\mathbb{R}^n_+) = \mathbb{R}^n_+$. Here we have chosen S to be diagonal and given by

$$S(v)_i = \begin{cases} e^{v_i}/e & \text{if } v_i > 1, \\ v_i & \text{if } v_i \in [-1, 1], \\ -e^{-v_i}/e & \text{if } v_i < -1. \end{cases}$$

Now define the differential equation

$$\dot{v} = g(v) = S'(S^{-1}(v))AS^{-1}(v). \tag{4}$$

Under a nonlinear change of coordinates (4) is just the system $\dot{z} = Az$, but we pretend not to know that. Instead we apply the algorithmic framework of the previous section to check that the origin is practically quasi-global asymptotically stable.

The Eaves-algorithm has been implemented (in MATLAB) as it is proposed in the paper [3] based on the K1 complex and using the integer labeling

$$l(v) = \min\{i \colon g(v) < -\varepsilon\}, \tag{5}$$

where ε is numerical design parameter and usually chosen very small, although it should be noted that larger ε give faster convergence. This algorithm is then applied to the simplex S_r and produces v_{KKM}. From here MATLAB's `ode45` is used to numerically compute the trajectory $\phi(t, v_{\mathrm{KKM}}, t \in [0, T]$ for some large $T > 0$ and to check that $\phi(T, v_{\mathrm{KKM}})$ is quite small.

As a proof of concept, numerical simulations have been performed on a single core of an Intel Core 2 Duo Processor operating at 2.4 GHz in MATLAB under Mac OS. The outcomes are shown in Table 1, giving run times, number of iterations and success rate of the Eaves-Algorithm, together with corresponding results obtained from the PQGAS algorithm.

	Eaves K1 algorithm		
n	time	#iter.	succ.rate
5	$0.11465s$	367.62	100%
10	$0.64855s$	2059.65	100%
15	$1.7833s$	5505.78	100%
25	$7.987s$	19742.84	100%

	PQGAS algorithm	
n	time	$\|\phi(T, v_{\mathrm{KKM}})\|_1$
5	$0.027537s$	$1.214 \cdot 10^{-06}$
10	$0.026541s$	$2.3631 \cdot 10^{-06}$
15	$0.025921s$	$3.3564 \cdot 10^{-06}$
25	$0.029049s$	$4.5434 \cdot 10^{-06}$

Table 1: Simulation results for $A \in \mathbb{R}_+^{n\times n}$, $A = -I + P$, $r = 10$, $T = 100$, $\alpha(A) = -0.2$, where matrix P described above is has uniformly distributed positive random entries, and 30% of these are set to zero. The numbers in each row are averages over 100 simulations.

The simulation shows that for any randomly chosen example the origin is indeed practically quasi-global asymptotically stable, with $\mathcal{R}$ and $\mathcal{C}$ given by the PQGAS algorithm.

7 Conclusions

We have demonstrated that comparison principles are quite powerful when they are combined with efficient numerical methods. It has been shown that large-scale system analysis can be done numerically, and comparison systems of order up to 25 are absolutely feasible.

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