Sum-separable Lyapunov functions for networks of ISS systems

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***Abstract*— It has been known for a decade now that for networks of input-to-state stable (ISS) systems a Lyapunov function for the network can be constructed as a maximum of re-scaled Lyapunov functions of the subsystems. In this work it is shown that —under the same conditions— instead one could as well choose a sum of re-scaled Lyapunov functions, which is not inherently non-smooth, thus answering a long open question. Moreover, our approach is constructive, and explicit formulas for the cases of two and three subsystems are stated.**

I. Introduction

We consider networks of n interconnected, input-to-state stable (ISS) systems and ask about stability properties of the composite system. In the case $n = 2$, this question is answered by the well known small-gain theorem [1] and its Lyapunov counterpart [2]. For $n > 2$, so-called generalised small gain theorems have been developed, initially for trajectory estimates, including [3], [4], [5], [6], [7], later for Lyapunov formulations of ISS, including [8], [9], [10], [11].

The Lyapunov constructions in these later works follow the original idea of [2] in defining a candidate Lyapunov function for the entire network as a maximum of (rescaled) known Lyapunov functions of the individual subsystems.

Orthogonal to these developments stood the question of how interconnections of systems satisfying more general stability notions than ISS could be handled, and what a suitable small-gain type condition should look like. For the notion of *integral* input-to-state stability, these developments were pioneered in [12], [13], [14], [15], [16]. A notable difference to the generalised small-gain theory is that these works have aggregated known Lyapunov functions of individual subsystems as a sum and not as a maximum into a Lyapunov function candidate for the composite system.

For a while it was unclear, whether the two approaches of aggregating individual Lyapunov functions were essentially equivalent or whether one was in some sense superior to the other. Eventually it was discovered in [17] that the maximum formulation of the composite Lyapunov function imposes restrictions on the stability types of the subsystems in the network. Much later, general existence and non-existence results and their implications have been addressed [18], [19], [20]. Notably, in [21] constructions of sum-type Lyapunov functions have been developed even for interconnections of integral ISS systems, but these appear quite technical and non-intuitive, thus limiting their possible application.

In this present work we propose a different construction that yields so-called sum-separable Lyapunov functions for networks of ISS systems. In the opinion of the authors, this present construction is more intuitive than the one of [21], as the composite Lyapunov function can be computed essentially directly from the gain functions of the network. However, our approach does not seem to translate to networks of integral input-to-state stable systems.

In a nutshell, our construction uses a gain operator $T(s) := \big(\bigoplus \gamma_{ij}(s_j)\big)_{i=1}^n$ defined on $\mathbb{R}^n_+$, where the functions γ_{ij} are either class $\mathcal{K}$ or zero and quantitatively encode the interconnection topology of the network. The symbol $\oplus$ denotes a (component-wise) maximum. Under the well-known small-gain condition the (computable) operator $T^*(s) := \bigoplus_{k=0}^\infty T^k(s)$ is well-defined and can be represented as a matrix with entries that are again class $\mathcal{K}$ functions or zero. Denoting these entries by t^*_{ij}, our Lyapunov function candidate is $V(x) := \sum_{i,j=1}^n t^*_{ij}\big(V_j(x_j)\big)$, where $V_j(x_j)$ denotes the individual Lyapunov functions evaluated for each individual subsystem. Clearly, V is sum-separable.

This paper is organised as follows: In the next section relevant notation is defined. The problem setup is described formally in Section III, once for interconnections of discrete-time systems and once for interconnections of continuous-time systems. Then the theoretical basis of this paper is developed in Section IV. Section V contains the main results regarding stability of interconnections of systems. In Section VI we explicitly state the formulas that yield Lyapunov functions for the cases of networks of two and three interconnected subsystems. Section VII concludes this paper and sets out directions for future work.

II. Notation

By $\mathbb{R}_+$ we denote the nonnegative real half-line $[0,\infty)$ and $\mathbb{R}^n_+$ is referred to as the nonnegative orthant. This nonnegative orthant is a cone and it induces a partial ordering on $\mathbb{R}^n_+$: Given $x, y \in \mathbb{R}^n$, we write $x \geq y$ if $x - y \in \mathbb{R}^n_+$, we write $x > y$ if $x \geq y$ and $x \neq y$, and we write $x \gg y$ if $x_i > y_i$ for all $i = 1, \ldots, n$. This partial order is called the *component-wise partial ordering* of the nonnegative orthant. Note that $x \not\geq y$, i.e., the logical negation of the first relation, means that at least one component of y is strictly greater than the corresponding component of x. We will make frequent use of the short hand notation $u \oplus v$ for $\max\{u, v\} := \big(\max\{u_1, v_1\}, \ldots, \max\{u_n, v_n\}\big)^T$. By $e_1, \ldots, e_n$ we denote the standard basis vectors of $\mathbb{R}^n$. The vector $(1, \ldots, 1)^T = \sum_i e_i \in \mathbb{R}^n$ is denoted by $\mathbf{1}$.

A function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and satisfies $\gamma(0) = 0$. A class $\mathcal{K}$ function is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. We will write

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$\gamma < \mathrm{id}$ for any function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\gamma(0) = 0$ and $\gamma(r) < r$ for all $r > 0$.

The composition of mappings T and S is denoted by $T \circ S$ and k-fold composition of a mapping with itself is denoted by $T^k = T \circ \ldots \circ T$. We make the convention that T^0 denotes the identity operation id on $\mathbb{R}^n_+$.

III. Problem setup

We consider networks of discrete-time systems and of continuous-time systems separately. In the first case the sum-separable Lyapunov function to be constructed in Lemma 4 of Section IV can essentially be applied in verbatim. In the continuous-time case the technical difficulty of linking the discrete-time Lyapunov function of Lemma 4 to a continuous-time dynamics has to be tackled. The simplest way to achieve this is to call upon a discrete-continuous comparison principle, as we dispense with for the purposes of this conference submission, since any other approach would come at the price of a significant overhead.

A. Discrete-time case

Given a network of n interconnected input-to-state stable systems

$$x_i^+ = f_i(x_1, \ldots, x_n, u), \quad i = 1, \ldots, n, \tag{1}$$

with $x_i \in \mathbb{R}^{N_i}$, $u \in \mathbb{R}^M$, we ask about the global asymptotic stability of the origin with respect to the composite system

$$x^+ = f(x, u) \tag{2}$$

where we denote the composite state vector by $x^T = (x_1^T, \ldots, x_n^T)$, when the disturbance input u is zero.

Here we say system (1) is *input-to-state stable* (ISS) if there exist functions γ_{ij} and δ of class $\mathcal{K}$ or equal to zero, $j = 1, \ldots, n$, functions $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$ and a continuously differentiable function $V_i\colon \mathbb{R}^{N_i} \to \mathbb{R}_+$ satisfying

$$\underline{\alpha}_i(\|x_i\|) \leq V(x) \leq \overline{\alpha}_i(\|x_i\|) \tag{3}$$

for all $x_i \in \mathbb{R}^{N_i}$ and

$$\begin{aligned} &V_i(f_i(x_1, \ldots, x_n, u)) \leq \\ &\quad \max\big\{\gamma_{i1}\big(V_1(x_1)\big), \ldots, \gamma_{in}\big(V_n(x_n)\big), \delta(\|u\|)\big\}. \end{aligned} \tag{4}$$

In the literature, this is called the *dissipative Lyapunov formulation* of ISS, and it is known to be equivalent to other notions. The functions V_i are called *ISS Lyapunov functions*. Observe that γ_{ii} should satisfy $\gamma_{ii} < \mathrm{id}$.

It is no loss of generality to assume that the gain δ from the external disturbance u to state x_i is the same for all subsystems $i = 1, \ldots, n$, or that the external disturbance u itself is the same for all interconnected subsystems.

Very clearly, the interconnection structure between the subsystems is encoded in the functions γ_{ij}. These functions induce a continuous mapping $T\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$, per

$$T(s) := \Big(\bigoplus \gamma_{ij}(s_j)\Big)_{i=1}^n \tag{5}$$

for all $s \in \mathbb{R}^n_+$. This mapping is monotone with respect to the component-wise ordering on $\mathbb{R}^n$, i.e., for vectors $u, v \in \mathbb{R}^n_+$ with $u \leq v$, we have $T(u) \leq T(v)$. In addition, the map T is *max-preserving* [22], [23], i.e., for $u, v \in \mathbb{R}^n_+$,

$$T(u \oplus v) = T(u) \oplus T(v).$$

B. Continuous-time case

Given a network of n interconnected input-to-state stable systems

$$\dot{x}_i = f_i(x_1, \ldots, x_n, u), \quad i = 1, \ldots, n, \tag{6}$$

with $x_i \in \mathbb{R}^{N_i}$, $u \in \mathbb{R}^M$, we ask about the global asymptotic stability of the origin with respect to the composite system

$$\dot{x} = f(x, u) \tag{7}$$

where we denote the composite state vector by $x^T = (x_1^T, \ldots, x_n^T)$, when u is equal to zero.

Here we say system (6) is *input-to-state stable* (ISS) if there exist functions γ_{ij} and δ of class $\mathcal{K}$ or equal to zero, $j = 1, \ldots, n$, functions $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$ and a continuously differentiable function $V_i\colon \mathbb{R}^{N_i} \to \mathbb{R}_+$ satisfying

$$\underline{\alpha}_i(\|x_i\|) \leq V(x) \leq \overline{\alpha}_i(\|x_i\|) \tag{8}$$

for all $x_i \in \mathbb{R}^{N_i}$ and

$$\frac{d}{dt} V_i(x_i(t)) = \nabla V_i(x_i) \cdot f_i(x_1, \ldots, x_n, u) < 0$$

whenever

$$V_i(x_i) > \max\big\{\gamma_{i1}\big(V_1(x_1)\big), \ldots, \gamma_{in}\big(V_n(x_n)\big), \delta(\|u\|)\big\}. \tag{9}$$

In the literature, this is called the *Lyapunov implication formulation* of ISS, and it is known to be equivalent to other notions. Again, γ_{ii} should satisfy $\gamma_{ii} < \mathrm{id}$. This time, one may safely assume that γ_{ii} is in fact zero, and again it is no loss of generality to assume that the gain δ from the external disturbance u to state x_i is the same for all subsystems $i = 1, \ldots, n$.

The interconnection structure between the subsystems is again encoded in the functions γ_{ij}, and these functions induce a continuous, monotone, max-preserving mapping of the form (5) as in the discrete-time case.

IV. Key technical facts

Before we state and prove our main results, we recall the small-gain condition and introduce a few key technical concepts.

The following *small-gain condition* is well known, and it comes in different guises.

Lemma 1: With T given by (5), the following properties are equivalent:

1) $T(s) \not\geq s$ for all $s \in \mathbb{R}^n_+$, $s > 0$.
2) $T(s) \geq s$ implies $s = 0$.
3) All cycles in T are contractions, i.e.,

$$\gamma_{i_1 i_2} \circ \ldots \circ \gamma_{i_k i_1} < \mathrm{id}$$

for all cycles.

4) All minimal cycles in T are contractions, i.e., those that do not contain smaller cycles.

5) For all $s \in \mathbb{R}^n_+$, $T^k(s) \to 0$ as $k \to \infty$.

These equivalences and a few more can be found in [24] and in [25]. The second formulation is taken from [22], [23] and its equivalence with the first formulation is obvious.

The next result follows immediately from writing out the respective definitions.

Lemma 2: Given two mappings $T, S \colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ of the form (5), then also the mappings $T \circ S$ and $T \oplus S$ $\big($the latter given by $(T \oplus S)(s) := T(s) \oplus S(s)\big)$ are of the form (5).

Applying this lemma repeatedly, we see that any finite composition $T^k = T \circ \ldots \circ T$ of mappings of the form (5) is again of the same form. However, passing to infinite compositions requires an additional assumption. We are mostly interested in the following fact.

Lemma 3: Let the mapping $T \colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be as in (5) and satisfy the small-gain condition of Lemma 1. Then $T^* \colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined as

$$T^*(s) := \bigoplus_{k=0}^{\infty} T^k(s) \tag{10}$$

for $s \in \mathbb{R}^n_+$ is again of the form (5).

Proof: Observe that due to the last property listed in Lemma 1 the supremum over all $k \geq 0$ is in fact a maximum over only finitely many terms. Each of these is of the form (5), so by repeated application of Lemma 2 the claim follows. ∎

For any T of the form (5) we can define an associated matrix of functions t_{ij} that are either of class $\mathcal{K}$ or zero and given by

$$t_{ij}(r) := T_i(re_j), \quad r \in \mathbb{R}_+.$$

With a slight abuse of notation we will denote the matrix $(t_{ij})_{i,j=1}^n$ again by T. If T is given by formula (5) then, of course, we have $t_{ij} = \gamma_{ij}$ for all i and j.

Lemma 4: Let the mapping $T \colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be as in (5) and satisfy the small-gain condition of Lemma 1, so that the $\mathcal{K}$ functions associated with T^* as in Lemma 3 is well-defined. Let

$$V(s) := \mathbf{1} \cdot \sum_{i=1}^{n} T^*(s_i e_i). \tag{11}$$

Then the continuous function $V \colon \mathbb{R}^n_+ \to \mathbb{R}_+$ enjoys the following properties:

1) There are class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}$ such that

$$\underline{\alpha}(\|s\|) \leq V(s) \leq \overline{\alpha}(\|s\|) \tag{12}$$

for all $s \in \mathbb{R}^n_+$.

2) For any $s \in \mathbb{R}^n_+$, $s \neq 0$, it holds that

$$V(T(s)) < V(s). \tag{13}$$

3) The function V is of the form

$$V(s) = \lambda_1(s_1) + \ldots + \lambda_n(s_n)$$

with functions λ_j of class $\mathcal{K}_\infty$ given by

$$\lambda_j(s_j) = \sum_i t^*_{ij}(s_j). \tag{14}$$

Proof: We start with the third assertion by rewriting (11) based on the matrix representation of T^* into

$$\begin{aligned} V(s) &:= \mathbf{1} \cdot \sum_{i=1}^{n} T^*(s_i e_i) \\ &= \sum_{ij} t^*_{ij}(s_j) \\ &= \sum_j \underbrace{\sum_i t^*_{ij}(s_j)}_{=: \lambda_j(s_j)} \end{aligned}$$

Clearly, the sum of class $\mathcal{K}$ functions is again of class $\mathcal{K}$. Since $t^*_{ii} \geq (T^0)_{ii} = \mathrm{id}$, each of the functions λ_j must indeed be unbounded and thus be of class $\mathcal{K}_\infty$.

From here it is clear that the function V is continuous. Now we establish (12). As noted before, clearly $T^*(s) \geq s$, and hence $V(s) \geq \mathbf{1} \cdot T^*(s) \geq \mathbf{1} \cdot s = \|s\|_1$, so that for the one-norm we may choose $\underline{\alpha} = \mathrm{id}$. To obtain an upper bound we define $\lambda^* := \bigoplus_{j=1}^n \lambda_j \in \mathcal{K}_\infty$ and then compute

$$\begin{aligned} V(s) &= \sum_j \lambda_j(s_j) \\ &\leq n\lambda^*(s_1 \oplus \ldots \oplus s_n) \\ &= n\lambda^*\big(\|s\|_\infty\big), \end{aligned}$$

so that we may choose $\overline{\alpha} = n\lambda^*$ for the max-norm. As all norms are equivalent on $\mathbb{R}^n$, these bounds can be converted into bounds for the Euclidean norm, if desired.

Finally, we tend to the second assertion and establish (13). Let $s \in \mathbb{R}^n_+$ be non-zero. Then we find that

$$\begin{aligned} V\big(T(s)\big) &= \mathbf{1} \cdot \sum_{i=1}^{n} T^*\big((T(s))_i e_i\big) \\ &= \mathbf{1} \cdot \sum_{i=1}^{n} \bigoplus_{k=0}^{\infty} T^k\big((T(s))_i e_i\big) \\ &= \mathbf{1} \cdot \sum_{i=1}^{n} \bigoplus_{k=1}^{\infty} T^k(s_i e_i) \\ &\leq \mathbf{1} \cdot \sum_{i=1}^{n} \bigoplus_{k=0}^{\infty} T^k(s_i e_i) \\ &= V(s). \end{aligned}$$

To establish that this inequality is actually a strict one, we note that the sum represented by $V(s)$ contains the terms $t^*_{ii}(s_i) = s_i$ for $i = 1, \ldots, n$ due to $T^0 = \mathrm{id}$ in the definition of T^*. In contrast, the sum represented by $V(T(s))$ is made up of terms containing the functions $(T^* \circ T)_{ij}$. Of these the off-diagonal entries are at most as large as the corresponding t^*_{ij}. The diagonal entries, however, are strictly smaller than id, as they consist of (a maximum of) cycles $t_{ij_1} \circ \ldots \circ t_{j_k i}$, which in turn must be contractions according to Lemma 1.

Now observe that due to $s \neq 0$ at least one s_i must be positive. Hence

$$\sum_i (T^* \circ T)_{ii}(s_i) < \sum_i t^*_{ii}(s_i) = \sum_i s_i,$$

so that in conclusion we have $V(T(s)) < V(s)$. ∎

V. Sum-separable Lyapunov function for networks of ISS systems

As explained in the beginning of Section III, we address the discrete-time and continuous-time problems separately.

A. Networks of discrete-time systems

Theorem 1: Let a network of interconnected systems (1) satisfying ISS estimates (3)–(4) be given. Suppose the interconnection gains γ_{ij} satisfy the small-gain condition. Denote the mapping induced by the gains by T as in (5) and let V be given by (11). Then the function $W(x) := V\big(V_1(x_1), \ldots, V_n(x_n)\big)$ defined for the composite network (2) is *sum-separable*, i.e., it can be written in the form

$$W(x) = \lambda_1\big(V_1(x_1)\big) + \ldots + \lambda_n\big(V_n(x_n)\big)$$

with suitable scaling functions $\lambda_i \in \mathcal{K}_\infty$. Moreover, there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ and $\delta^* \in \mathcal{K}$ such that for all $x \in \mathbb{R}^N$,

$$\underline{\alpha}(\|x\|) \le W(x) \le \overline{\alpha}(\|x\|) \tag{15}$$

and for all $x \in \mathbb{R}^N$, $x \neq 0$, and all $u \in \mathbb{R}^M$,

$$W(f(x,u)) < W(x) + \delta^*(\|u\|). \tag{16}$$

In particular, the composite system (2) is 0-GAS (that is, the origin is GAS for $u \equiv 0$). Furthermore, the composite system satisfies the *bounded-inputs-bounded-states property* (which explains itself).

Remark 1: The statement of the theorem is slightly too weak: It is known that under the given assumptions the network must be ISS from u to x, cf. [26]. However, our present sum-separable Lyapunov function does not establish this stronger fact. For W to be an ISS Lyapunov function, instead of (16) we would need an estimate like

$$W(f(x,u)) \le \alpha\big(W(x)\big) + \delta^*(\|u\|)$$

with a class $\mathcal{K}_\infty$ function α satisfying

$$(\alpha + \rho) < \mathrm{id}$$

for yet another class $\mathcal{K}_\infty$ function. This seemingly technical construction ensures that W decreases despite the presence of large disturbances $\|u\|$, as long as $W(x)$ is sufficiently large.

Proof: The sum-separable form of W follows immediately from the sum-separability of the function V in Lemma 4. Similarly, (15) follows directly from (12). In order to establish (16), we start by observing that any class $\mathcal{K}$ function is max-preserving, i.e., for $\gamma \in \mathcal{K}$ (or $\gamma \equiv 0$ for that matter), we have

$$\gamma(a \oplus b) = \gamma(a) \oplus \gamma(b)$$

for all numbers $a, b \in \mathbb{R}_+$. Now, $V(s) = \sum_{i,j} t^*_{ij}(s_j)$ is a sum of such functions. Hence, for vectors $a, b \in \mathbb{R}^n_+$ we have

$$\begin{aligned} V(a \oplus b) &= \sum_{i,j} t^*_{ij}(a_j \oplus b_j) \\ &= \sum_{i,j} \big(t^*_{ij}(a_j) \oplus t^*_{ij}(b_j)\big) \\ &\le \sum_{i,j} t^*_{ij}(a_j) + \sum_{i,j} t^*_{ij}(b_j) \\ &= V(a) + V(b). \end{aligned}$$

Re-writing estimates (4) into one vector-inequality using the short-hand notation $v_i := V_i(x_i)$ and $v_i^+ := V_i(f_i(x_1, \ldots, x_n, u))$, we obtain

$$v^+ \le T(v) \oplus \delta(\|u\|)\mathbf{1}.$$

Using that $x \neq 0$ implies $v \neq 0$, this leads us to

$$\begin{aligned} W(f(x,u)) &= V(v^+) \\ &\le V\big(T(v) \oplus \delta(\|u\|)\mathbf{1}\big) \\ &\le V\big(T(v)\big) + V\big(\delta(\|u\|)\mathbf{1}\big) \\ &< V(v) + V\big(\delta(\|u\|)\mathbf{1}\big) \\ &= W(x) + \delta^*(\|u\|) \end{aligned}$$

whenever $x \neq 0$, where $\delta^*(r) := V\big(\delta(r)\mathbf{1}\big)$ is a function of the same class as the function δ. This previous estimate establishes global asymptotic stability of the origin for the case that the input disturbance u is identically zero. As a side note, it also establishes *local* (sometimes referred to as practical) input-to-state stability, that is input-to-state stability for states and inputs confined to small compact sets.

Now assume $v \ge \delta(\|u\|)\mathbf{1}$, i.e., that the input disturbances are bounded and, in comparison, states of the interconnected subsystems are large. In this case we have $v^+ \le T(v) \oplus \delta(\|u\|)\mathbf{1} \le T(v) \oplus v$, so $T^*(v^+) \le T^*(T(v)) \oplus T^*(v) = T^*(v)$. This in turn implies $W(x^+) \le W(x)$ whenever $\max_i V_i(x_i) \ge \|u\|$. Hence, any bounded disturbance input u results in a bounded state trajectory. ■

B. Networks of continuous-time systems

The very function V of Lemma 4 can be used to establish stability properties of a network of continuous-time ISS systems. Here we dispense with 0-GAS, that is global asymptotic stability of the interconnection (7) when the input u is set to zero.

Theorem 2: Let a network of interconnected systems (6) satisfying ISS estimates (8)–(9) be given. Suppose the interconnection gains γ_{ij} satisfy the small-gain condition. Denote the mapping induced by the gains by T as in (5) and let V be given by (11). Then the function V establishes 0-GAS of the composite system (7).

Proof: Observe that V is a Lyapunov function for the discrete-time dynamics

$$s^+ = T(s)$$

on $\mathbb{R}^n_+$. By a comparison principle along the lines of [27, Corollary IV.2], this implies 0-GAS of the interconnection (7). ■

$$T^* = \mathrm{id}_{\mathbb{R}^3_+} \oplus T \oplus T^2 = \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix} \oplus \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \oplus \tag{19}$$

$$\begin{pmatrix} \gamma_{11}^2 \oplus \gamma_{12} \circ \gamma_{21} \oplus \gamma_{13} \circ \gamma_{31} & \gamma_{11} \circ \gamma_{12} \oplus \gamma_{12} \circ \gamma_{22} \oplus \gamma_{13} \circ \gamma_{32} & \gamma_{11} \circ \gamma_{13} \oplus \gamma_{12} \circ \gamma_{23} \oplus \gamma_{13} \circ \gamma_{33} \\ \gamma_{11} \circ \gamma_{21} \oplus \gamma_{21} \circ \gamma_{22} \oplus \gamma_{23} \circ \gamma_{31} & \gamma_{12} \circ \gamma_{21} \oplus \gamma_{22}^2 \oplus \gamma_{23} \circ \gamma_{32} & \gamma_{13} \circ \gamma_{21} \oplus \gamma_{22} \circ \gamma_{23} \oplus \gamma_{23} \circ \gamma_{33} \\ \gamma_{11} \circ \gamma_{31} \oplus \gamma_{21} \circ \gamma_{32} \oplus \gamma_{31} \circ \gamma_{33} & \gamma_{12} \circ \gamma_{31} \oplus \gamma_{22} \circ \gamma_{32} \oplus \gamma_{32} \circ \gamma_{33} & \gamma_{13} \circ \gamma_{31} \oplus \gamma_{23} \circ \gamma_{32} \oplus \gamma_{33}^2 \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{id} & \gamma_{12} \oplus \gamma_{13} \circ \gamma_{32} & \gamma_{13} \oplus \gamma_{12} \circ \gamma_{23} \\ \gamma_{21} \oplus \gamma_{23} \circ \gamma_{31} & \mathrm{id} & \gamma_{23} \oplus \gamma_{13} \circ \gamma_{21} \\ \gamma_{31} \oplus \gamma_{21} \circ \gamma_{32} & \gamma_{32} \oplus \gamma_{12} \circ \gamma_{31} & \mathrm{id} \end{pmatrix} \tag{20}$$

VI. Explicit formulæ

A. Two subsystems

For two interconnected subsystems we have a gain matrix

$$T = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$$

with $\gamma_{ij} \in (\mathcal{K} \cup \{0\})$. The associated max-separable mapping $T\colon \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is given by

$$T(s_1, s_2) = \begin{pmatrix} \max \{\gamma_{11}(s_1), \gamma_{12}(s_2)\} \\ \max \{\gamma_{21}(s_1), \gamma_{22}(s_2)\} \end{pmatrix}.$$

The small-gain condition requires that all of the following conditions are satisfied:

$$\gamma_{11} < \mathrm{id}$$
$$\gamma_{22} < \mathrm{id}$$

and

$$\gamma_{12} \circ \gamma_{21} < \mathrm{id} \tag{17}$$

Note that (17) holds if and only if

$$\gamma_{21} \circ \gamma_{12} < \mathrm{id}$$

holds. This can be seen by observing that every $\mathcal{K}_\infty$ function has an inverse which is again a $\mathcal{K}_\infty$ function. And without loss of generality, we may assume that all the γ_{ij} are in fact of class $\mathcal{K}_\infty$.

Writing $s = (s_1, s_2)^T$ and under the small-gain assumption we proceed to compute

$$\begin{aligned} T^*(s) &= \bigoplus_{k=0}^{\infty} T^k(s) \\ &= s \oplus T(s) \\ &= \begin{pmatrix} \mathrm{id} & \gamma_{12} \\ \gamma_{21} & \mathrm{id} \end{pmatrix} \end{aligned} \tag{18}$$

as already

$$T^2 = \begin{pmatrix} \gamma_{11}^2 \oplus \gamma_{12} \circ \gamma_{21} & \gamma_{12} \circ \gamma_{22} \oplus \gamma_{11} \circ \gamma_{12} \\ \gamma_{21} \circ \gamma_{11} \oplus \gamma_{22} \circ \gamma_{21} & \gamma_{22}^2 \oplus \gamma_{21} \circ \gamma_{12} \end{pmatrix}$$

is, due to the small-gain condition, component-wise less than the matrix $\mathrm{id}_{\mathbb{R}^2_+} \oplus T$ computed above.

It is worth noting that, basically, the (i, j)th entry of the matrix T^* consists of the maximum over all possible paths from node j to node i in the weighted graph with n vertices and directed edges weighted with the functions γ_{ij}. Because any path longer than n edges will contain a cycle, the infinite supremum in the definition of T^*, cf. (10), is in fact a maximum over at most n powers of T.

From (18) we obtain

$$\lambda_1(s_1) = t^*_{11}(s_1) + t^*_{21}(s_1) = s_1 + \gamma_{21}(s_1)$$

and

$$\lambda_2(s_2) = t^*_{12}(s_2) + t^*_{22}(s_2) = s_2 + \gamma_{12}(s_2),$$

yielding

$$V(s) = (\mathrm{id} + \gamma_{21})(s_1) + (\mathrm{id} + \gamma_{12})(s_2).$$

Notably, this function is as smooth as the gains between the two subsystems.

B. Three subsystems

For three interconnected subsystems, respectively, in the case $n = 3$, the computations follow the same steps as in the previous subsection. As argued before, from

$$T = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}$$

and under the assumption that all cycles are contractions, we can compute T^* simply as in (19)–(20), where we note that the simplifications in (20) are possible because all cycles are contractions.

From (20) we obtain

$$\begin{aligned} \lambda_1 &= \mathrm{id} + \gamma_{21} \oplus \gamma_{23} \circ \gamma_{31} + \gamma_{31} \oplus \gamma_{21} \circ \gamma_{32} \\ \lambda_2 &= \gamma_{12} \oplus \gamma_{13} \circ \gamma_{32} + \mathrm{id} + \gamma_{32} \oplus \gamma_{12} \circ \gamma_{31} \\ \lambda_3 &= \gamma_{13} \oplus \gamma_{12} \circ \gamma_{23} + \gamma_{23} \oplus \gamma_{13} \circ \gamma_{21} + \mathrm{id}. \end{aligned}$$

Very clearly, $V(s) = \lambda_1(s_1) + \lambda_2(s_2) + \lambda_3(s_3)$ is generally less smooth than its counterpart for $n = 2$, as every function λ_i contains maxima of other functions.

However, a scalar function of one scalar argument is much easier to approximate by a smooth function than a function of several arguments.

VII. Conclusions and future directions

In this work sum-separable Lyapunov functions for networks of ISS systems have been proposed. Their advantage is that they can be constructed systematically from the knowledge of the interconnection gains of the network.

For the case of two interconnected subsystems the present construction may yield a smooth Lyapunov function, at least as long as the involved gains are smooth. For larger networks additional smoothing steps could be applied to the individual scaling functions. It remains to be seen how successful such an approach will be in applications.

A shortcoming of the present work in comparison to the existing literature on max-separable Lyapunov functions is that at this point the theory does not provide Lyapunov functions for ISS, but only for 0-GAS. At this point the authors believe that the sharpness of some estimates may be improved to successfully address this issue. Another direction to address in the future centres about the lifting of the present results to networks of integral input-to-state stable systems, where other sum-separable constructions for Lyapunov functions are available in the literature.

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