# Computing asymptotic gains of large-scale interconnections

Björn S. Rüffer    Hiroshi Ito    Peter M. Dower

***Abstract*— This paper considers the problem of verifying stability of large-scale nonlinear dynamical systems. Using a comparison principle approach we present a numerical method of estimating the asymptotic gain characterizing the effect of external disturbances on the stability of a large-scale interconnection. The unique idea is to make use of solely the knowledge of one single trajectory of the comparison system for estimating the behavior of all possible trajectories. It is shown that an asymptotic gain can be obtained from just a single trajectory of a disturbance-free comparison system. The single-trajectory approach leads to a computationally cheap implementation with which we can numerically check whether or not a large-scale system is input-to-state practically stable.**

## I. Introduction

We consider large-scale interconnections of input-to-state practically stable (ISpS) systems in an arbitrary interconnection topology. It is known that under suitable conditions such interconnections yield an ISpS composite system. The difficult part here is to verify the stability condition, also known as generalized small-gain condition. This paper demonstrates that this task can be supported by following the these few steps:

i) Consider the comparison system

$$\dot{v} = g(v) + w, , \quad v, w \in \mathbb{R}^n_+, \tag{1}$$

of the large-scale interconnection subject to external disturbances. This is given.

ii) Compute one special trajectory of the autonomous system

$$\dot{v} = g(v) \tag{2}$$

(or a sampled version of it).

iii) Estimate the stability of the large-scale interconnection with respect to the disturbances by evaluating the right hand side of (2) only along the previously computed trajectory.

Hence, computing just one special trajectory of the comparison system numerically leads us to an asymptotic gain of the large-scale interconnection. In other words, this paper presents an analytical formula for the asymptotic gain in terms of the single special trajectory as well as theoretical tools justifying such an approach.

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B. S. Rüffer and P. M. Dower are with the Department of Electrical and Electronic Engineering, University of Melbourne, Parkville, VIC 3010, Australia.

H. Ito is with the Department of Systems Design and Informatics, Kyushu Institute of Technology, 680-4 Kawazu, Iizuka, Fukuoka 820-8502, Japan.

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Stability conditions for large-scale interconnections of dynamical systems have been studied at least since the 1960s, and there is a significant amount of literature on the subject. Among other approaches, most notable for our purposes are vector Lyapunov function and comparison principle approaches (e.g., [14]). More recently, stability of large-scaled systems has been studied within the input-to-state stability (ISS) framework [5], [7], [12], [13] including input-output stability and systems with time delay. These approaches lead to generalized small-gain conditions, based on the requirement that a particular operator is a contraction in a "strict" way. Some of these small-gain conditions are analytically feasible to handle, especially the so-called cyclic or max-type small-gain conditions [7], [15]. Here one has to verify that a couple of functions are strictly smaller than the identity. Analytically more difficult to verify are other small-gain conditions, especially those where individual gains enter subsystem estimates in forms of sums, e.g., as in [5], [17]. This motivated an efficient scheme to verify these conditions on prescribed regions numerically, [6]. A computational version of a comparison principle for global asymptotic stability has been pursued in [18]. Numerical verification of the ISS property and computation of transient and asymptotic bounds has been a practically important issue. For example, a dynamic programming approach has been proposed in [10] for obtaining tight asymptotic gains of ISS systems. In this paper, we pursue a way to compute asymptotic gains of large-scale interconnections based on information about only individual systems.

In stability analysis of dynamical systems monotonicity is a useful property [1], [2], [9], [16], which is sometimes possessed by either original systems or their comparison systems. In this paper we also make use of the monotonicity. However, our unique idea is to utilize the monotonicity to numerically verify stability, which lies in just one trajectory. Here we provide a method to test whether or not a large-scale system is ISpS by computing the asymptotic bound on the states for a given magnitude of input disturbance numerically.

Our results are based on the following reasoning: Consider a monotone system (1) with $g\colon \mathbb{R}^n_+ \to \mathbb{R}^n$ locally Lipschitz and quasi-monotone nondecreasing, $g(0) = 0$, and $v, w \in \mathbb{R}^n_+$. This system evolves on the non-negative orthant $\mathbb{R}^n_+$. If this system is input-to-state stable from $w$ to $v$ then the autonomous part (2) admits a special trajectory $\tilde{\phi}(t)$ (or stable manifold) defined for all times, along which $g(\tilde{\phi}(t)) \ll 0$, i.e., here the vector field is strictly negative in all its components. Knowledge of this trajectory essentially reduces the system dynamics down to one dimension. Furthermore, the magnitude of the vector field $g$ along this trajectory provides a bound on the maximal input disturbance $\overline{w}(t)$ so that $g(\tilde{\phi}(t)) + \overline{w}(t) \leq 0$ (this will be made precise later). From here, an asymptotic gain of the original system (1) subject to the disturbance $w$ is simple to compute, and via the vector Lyapunov function the asymptotic gain can be translated to the large-scale system.

The paper is organized as follows. First we introduce necessary notation. Then, in Section III we recall some facts on monotone systems, which will in our context appear as comparison systems. Section IV introduces the large-scale interconnection of dynamical systems together with the corresponding comparison system. The main results are Theorem 4.8 and Corollary 5.1, which is especially tailored for numerical implementation. Examples and numerical statistics are provided in Section VI, with a conclusion provided in Section VII.

## II. Notation

The set $\mathbb{R}_+$ denotes the non-negative real numbers, $\mathbb{R}^n_+$ is called the *positive orthant* in $\mathbb{R}^n$ (non-negative orthant would be more accurate though), and it defines the following component-wise partial order: Vectors $v, y \in \mathbb{R}^n$ are ordered by $v \leq y$ if $y - v \in \mathbb{R}^n_+$, $v \ll y$ if $y - v \in \operatorname{int} \mathbb{R}^n_+$, where $\operatorname{int} A$ denotes the interior of a set $A$, and $v < y$ if $v \leq y$ and $v \neq y$. If $v, w \in \mathbb{R}^n$, $v \leq w$, then the compact set $[v, w] = \{y \in \mathbb{R}^n : v \leq y \leq w\}$ denotes an order interval. Denote by $x - \mathbb{R}^n_+$ the set $\{y \in \mathbb{R}^n : y \leq x\}$ and by $x + \mathbb{R}^n_+$ the set $\{y \in \mathbb{R}^n : y \geq x\}$.

We also introduce the following component-wise versions of absolute value and norm: For vectors $v \in \mathbb{R}^n$ we write $\underline{|v|} := \max\{v, -v\} \in \mathbb{R}^n_+$. For $x \in \mathbb{R}^{N_1 \times \dots \times \mathbb{R}^{N_n}}$ we write $\underline{\|x\|} = (\|x_1\|, \dots, \|x_n\|)^T$. The choice of norm $\|\cdot\|$ is arbitrary, but sometimes we use the 1-norm which is denoted by $\|\cdot\|_1$.

For maps $f$ and $g$, $(f \circ g)(x)$ denotes $f(g(x))$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if $\gamma(0) = 0$, $\gamma$ is continuous and strictly increasing. A function is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded.

Given a compact set $\mathcal{A} \in \mathbb{R}^N$ by $\|x\|_\mathcal{A}$ we denote the distance of $x \in \mathbb{R}^N$ to $\mathcal{A}$, i.e., $\|x\|_\mathcal{A} = \inf_{y \in \mathcal{A}} \|x - y\|$. Although the symbol might suggest otherwise, the distance function with respect to the set $\mathcal{A}$, $\|\cdot\|_\mathcal{A}$, is not a norm. If $\mathcal{A} = \mathcal{A}_1 \times \ldots \times \mathcal{A}_n$ we also write $\underline{\|x\|}_\mathcal{A} = (\|x_1\|_{\mathcal{A}_i}, \dots, \|x_n\|_{\mathcal{A}_i})^T$, and similarly for $L_\infty$ norms.

## III. Preliminaries

Here we collect a few facts regarding monotone systems. We refer the reader to [14], [19], [20].

A map $g : \mathbb{R}^n \to \mathbb{R}^n$ is *quasi-monotone nondecreasing* if $g_i(v) \leq g_i(w)$ whenever $v \leq w \in \mathbb{R}^n$ and $v_i = w_i$. This property is also known under a different name, namely that $g$ is of "type K". If $g$ is locally Lipschitz continuous, then sufficient for $g$ to be quasi-monotone nondecreasing is that for almost all $v$, $\frac{\partial g_i}{\partial v_j}(v) \geq 0$.

*Lemma 3.1 (Facts; monotone systems):* Consider

$$\dot{v} = g(v) \tag{3}$$

with $g(0) = 0$, $g \colon \mathbb{R}^n_+ \to \mathbb{R}^n$ locally Lipschitz and quasi-monotone nondecreasing. Then

i) Trajectories of (3) are constrained to $\mathbb{R}^n_+$ as long as they exist.
ii) (Ordering of solutions). The flow of (3) is order preserving: For $\prec$ denoting one of $\leq, <, \ll$, if $v^0 \prec w^0$ then for all times $t$ where both solutions exist $\phi(t, v^0) \prec \phi(t, w^0)$. Here, $v^0$ and $w^0$ denote the initial conditions.
iii) The ordering of solutions remains true if (3) is replaced by the time varying system
$$\dot{v} = g(t, v), \tag{4}$$
where $g(t, \cdot)$ quasi-monotone nondecreasing for almost all $t$.
iv) The ordering of solutions even remains true (with the ordering as above) if $\phi_v(t, t^0, v^0)$ is a trajectory of the system $\dot{v} = g(t, v)$, $\phi_w(t, t^0, w^0)$ is a trajectory of the system $\dot{w} = \tilde{g}(t, w)$, and $\tilde{g}(t, \cdot) - g(t, \cdot)$ is quasi-monotone nondecreasing for almost all $t \in \mathbb{R}$.
v) If the origin is attractive with respect to (3) with region of attraction $\mathcal{B}$ then
   - for all $v \in \mathcal{B}, v \neq 0$, $g(v) \not\geq 0$;
   - for all $r > 0$ such that $S_r := \{v \in \mathbb{R}^n_+ : \|v\|_1 = r\} \subset \mathcal{B}$,
   $$\Omega(g) \cap S_r \neq \emptyset,$$
   where $\Omega(g) := \{v \in \mathbb{R}^n_+ : g(v) \ll 0\}$;
   - the origin is stable in the sense of Lyapunov.
vi) The set $\Omega$ defined above is forward invariant.
vii) If the origin is globally attractive with respect to (3) then there exists a trajectory $\tilde{\phi}$, defined for all times, along which $g(\tilde{\phi}(t)) \ll 0$.

## IV. Large-scale interconnections

### A. Comparison principle

Let interconnected systems

$$\dot{x}_i = f_i(x, u_i) \tag{5}$$

be given, with $f(x, u) := (f_1(x, u_1)^T, \dots, f_n(x, u_n^T)^T$ locally Lipschitz in $x$, uniformly for $u$ in compact sets; $x_i \in \mathbb{R}^{N_i}$, $u_i \in \mathbb{R}^{M_i}$, $i = 1, \dots, n$, with $x = (x_1^T, \dots, x_n^T)^T$ and $u = (u_1^T, \dots, u_n^T)^T$. The interconnection of systems (5) yields the composite, large-scale system

$$\dot{x} = f(x, u). \tag{6}$$

We are interested in stability of a compact set $\mathcal{A}_i$ with respect to system (5). Without loss of generality, $0 \in \mathcal{A}_i$, and as a special case we have $\mathcal{A}_i = \{0\}$.

Assume that $n$ smooth functions $V_i \colon \mathbb{R}^{N_i} \to \mathbb{R}_+$, $i = 1, \dots, n$ are given, satisfying for some $\psi_{1i}, \psi_{2i} \in \mathcal{K}_\infty$, $\alpha_i \in \mathcal{K}_\infty$, and $\gamma_{ij}, \eta_i \in (\mathcal{K} \cup \{0\})$,

$$\psi_{1i}(\|x_i\|_{\mathcal{A}_i}) \leq V_i(x_i) \leq \psi_{2i}(\|x_i\|_{\mathcal{A}_i}) \tag{7}$$

as well as the dissipation inequalities

$$\langle \nabla V_i(x_i), f_i(x,u_i)\rangle \leq \\ -\alpha_i(V_i(x_i)) + \sum_j \gamma_{ij}(V_j(x_j)) + \eta_i(\|u_i\|) \quad (8)$$

for all $x$ and $u_i$. The function $V_i$ is referred to as an ISS Lyapunov function of system (5) with respect to $\mathcal{A}_i$ [22]. This notion is called *input-to-state stability with respect to a compact set* $\mathcal{A}_i$, and it is equivalent to input-to-state *practical* stability (ISpS), cf. [11]. In case that $\mathcal{A}_i = \{0\}$ this notion of course reduces to the familiar input-to-state stability (ISS) [21].

We will assume further that the functions $\alpha_i$ and $\gamma_{ij}$ are all locally Lipschitz. This assumption can, however, be relaxed, but that is not subject of this paper.

We will use the shorthand notation $\underline{V}(x) = (V_1(x_1),\dots,V_n(x_n))^T$. The map $\underline{V}\colon \mathbb{R}^N \to \mathbb{R}^n_+$ is sometimes called a *vector Lyapunov function* [14].

The dissipation inequalities (8) lead to a *comparison system*

$$\dot v = g(v) + \eta(w), \quad v,w \in \mathbb{R}^n_+, \quad (9)$$

with $g_i(v) = -\alpha_i(v_i) + \sum_j \gamma_{ij}(v_j)$, and $\eta(w) = (\eta_1(w_1),\dots,\eta_n(w_n))^T$. The function $g$ is obviously quasi-monotone nondecreasing. Moreover, since the $\alpha_i$ and $\gamma_{ij}$ are locally Lipschitz, so is $g\colon \mathbb{R}^n_+ \to \mathbb{R}^n$. By Lemma 3.1, the dynamics of (9) is confined to $\mathbb{R}^n_+$, and solutions are ordered. Hence, the comparison system (9) is a *monotone system*.

The comparison system (9) and the composite system (6) arising as the interconnection of individual subsystems (5) are related in the expected way. Denote $\mathcal{A} = \prod_{i=1}^n \mathcal{A}_i$. We cite the following result from [19].

*Theorem 4.1 (Comparison principle):* i) If system (9) is ISS from $w$ to $v$ then system (6) is ISS from $u$ to $x$ with respect to the compact set $\mathcal{A}$.

ii) If $L : \mathbb{R}^n_+ \to \mathbb{R}_+$ is an ISS Lyapunov function for (9) then $V(x) := L(\underline{V}(x))$ is an ISpS Lyapunov function for (6) (i.e., an ISS Lyapunov function with respect to $\mathcal{A}$).

This theorem motivates that it is enough to understand the comparison system in order to verify not only internal stability of the original large-scale system, but also the stability in the presence of external disturbances.

### B. ISS of the monotone system

The reasoning behind the main result, Theorem 4.8 in Section IV-C, relies on the following technical facts. Proofs are given in the appendix.

*Lemma 4.2:* Consider a dynamical system $\dot x = f(x)$, $f\colon \mathbb{R}^n \to \mathbb{R}^n$ quasi-monotone nondecreasing, locally Lipschitz, forward complete (i.e., solutions exist for all times), $f(\overline{x}) \leq 0$ and for all $x > \overline{x}$, $f(x) \not\geq 0$. Assume for all $x > \overline{x}$ there exists a $y \geq x$ so that $f(y) \ll 0$. Then all solutions starting in $\overline{x} + \mathbb{R}^n_+$ are attracted to the set $\overline{x} - \mathbb{R}^n_+$.

Conversely, if $f$ is forward complete and quasi-monotone nondecreasing, $f(\overline{x}) \leq 0$, and all solutions starting in $\overline{x}+\mathbb{R}^n_+$ are attracted to the set $\overline{x}-\mathbb{R}^n_+$, then necessarily for all $x > \overline{x}$, $f(x) \not\geq 0$.

*Lemma 4.3:* Assume that $g$ is quasi-monotone nondecreasing and locally Lipschitz with $g(0) = 0$.

i) System (9) is ISS from $w$ to $v$ if and only if

$$\dot v = g(v) + w \quad (10)$$

is ISS from $w$ to $v$.

ii) System (10) is ISS if and only if for every $v, w \in \mathbb{R}^n_+$ there exists a $\overline{v} \geq v$, so that $g(\overline{v}) + w \ll 0$.

*Lemma 4.4:* Assuming that (10) is ISS, then

i) there exists a special solution $\tilde\phi(\cdot)$ of the autonomous dynamics

$$\dot v = g(v), \quad (11)$$

which is defined for all times and $g(\tilde\phi(t)) \ll 0$ for all $t \in \mathbb{R}$.

ii) Let $\mathcal{O} := \{\tilde\phi(t)\}_{t\in\mathbb{R}} \cup \{0\}$, a closed set. This set can be parametrized as by path $\sigma\colon \mathbb{R}_+ \to \mathcal{O}$ satisfying $\|\sigma(r)\|_1 = r$ for all $r \geq 0$.

iii) Each component $\sigma_i$ of this path is a class $\mathcal{K}$ function, at least one of them is unbounded.

By utilizing the previous result, Lemma 4.3 can be made more precise; often we can find parametrized versions of $v$ and $w$ such that $g(v) + w \ll 0$, as follows.

*Lemma 4.5:* Assume that $g$ is quasi-monotone nondecreasing and locally Lipschitz with $g(0) = 0$. Let (10) be ISS from $w$ to $v$ and $\sigma$ be given by Lemma 4.4. Assume that $\sigma_i$ is of class $\mathcal{K}_\infty$, $i = 1,\dots,n$. If the map $\tilde\rho\colon \mathbb{R}_+ \to \mathbb{R}^n_+$ given by $\tilde\rho(r) := -g(\sigma(r))$ satisfies $\tilde\rho(r) \gg 0$ for all $r > 0$ and

$$\liminf_{r\to\infty} \tilde\rho_i(r) = \infty$$

then there exist functions $\rho_i \in \mathcal{K}_\infty$ such that $\rho_i(r) < \tilde\rho_i(r)$ for all $r > 0$. In particular,

$$g(\sigma(r)) + \rho(r) \ll 0 \text{ for all } r > 0.$$

A related converse statement of Lemma 4.5 is also true:

*Lemma 4.6:* Assume that $g$ is quasi-monotone nondecreasing and locally Lipschitz with $g(0) = 0$. If there exist maps $\sigma, \rho\colon \mathbb{R}_+ \to \mathbb{R}^n_+$ with $\sigma_i, \rho_i \in \mathcal{K}_\infty$ such that $g(\sigma(r)) + \rho(r) \ll 0$ for all $r > 0$ then (10) is ISS from $w$ to $v$.

These are the necessary ingredients to get the main result in the next section.

*Remark 4.7:* In view of Lemma 4.6 it may seem that $\rho$ and $\sigma$ always have to be of class $\mathcal{K}_\infty$ in every component. This is, however, not true. For example, the system

$$\dot v = g(v) + w$$

with $g\colon \mathbb{R}^2_+ \to \mathbb{R}^2$ given by

$$g(v) = \begin{pmatrix} -v_1 + \frac{1}{2}v_2 \\ -v_2 + \tanh(v_1) \end{pmatrix}$$

admits paths $\sigma$, with $g(\sigma(r)) \ll 0$ for $r > 0$, that are bounded in the second component. In this case the resulting $\rho$ is also bounded in the second component.

### C. Main result

*Theorem 4.8:* Let a large-scale system (6), decomposed into subsystems (5) satisfying (7)–(8), be given. Assume that $g$ arising from (8) is quasi-monotone nondecreasing and locally Lipschitz with $g(0) = 0$.

Assume there exist functions $\sigma, \rho\colon \mathbb{R}_+ \to \mathbb{R}^n_+$ with $\sigma_i, \rho_i \in \mathcal{K}_\infty$, $i = 1, \ldots, n$, such that $g(\sigma(r)) + \rho(r) \ll 0$. Then the gain of the magnitude of external disturbances to the asymptotic magnitude of the states of the nominal system (6) is component-wise bounded from above by a map $G\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ given by

$$G(w) = \underline{\psi_1^{-1}} \circ \sigma \circ \max_i \rho_i^{-1} \circ \eta_i(w_i), \tag{12}$$

where $\underline{\psi_1^{-1}}(v) := (\psi_{11}^{-1}(v_1), \ldots, \psi_{1n}^{-1}(v_n))^T$ (which are given by (7)), in the sense that

$$\limsup_{t\to\infty} \underline{\|}x(t, x^0, u(t))\underline{\|}_{\mathcal{A}} \leq G(\underline{\|}u\underline{\|}_{L_\infty}). \tag{13}$$

*Remark 4.9:* We may assume that $\sigma$ is parametrized such that $\|\sigma(r)\|_1 = r$ for all $r \geq 0$. For any other parametrization of $\sigma$, e.g., $\sigma \circ \kappa$ with $\kappa \in \mathcal{K}_\infty$ would have led to the same estimate, since in this case one has $\rho \circ \kappa$ instead of $\rho$, and the $\kappa$ will eventually cancel out in (12).

*Proof.* We will only show that the asymptotic gain of the comparison system (10) is given by

$$\limsup_{t\to\infty} v(t) \leq \sigma \circ \max_i \rho_i^{-1}(\text{ess. sup}_t\, w_i(t)). \tag{14}$$

From here the result follows by utilizing the definitions of $V_i$, $\psi_{1i}$ and $\eta_i$, for $i = 1, \ldots, n$.

Take $w(\cdot) \in L_\infty$ and denote $\overline{w} = \text{ess. sup}_t\, w(t)$. Since $\rho_i \in \mathcal{K}_\infty$, $i = 1, \ldots, n$, there exists a minimal $\overline{r} \in \mathbb{R}_+$ such that $\rho(\overline{r}) \geq \overline{w}$, and this is $\overline{r} = \max_i \rho_i^{-1}(\overline{w}_i)$. In other words, $w(\cdot) \in [0, \rho(\overline{r})]$ for almost all $t \in \mathbb{R}$.

Now we compare solutions of the following three systems,

$$\dot{v}(t) = g(v(t)) + w(t) \tag{15}$$

$$\dot{v}(t) = g(v(t)) + \overline{w} \tag{16}$$

$$\dot{v}(t) = g(v(t)) + \rho(\overline{r}). \tag{17}$$

For a fixed initial conditions, the solutions to these systems are ordered in the following way, for all times $t \geq t^0$,

$$\phi_{(15)}(t, t^0, v^0) \leq \phi_{(16)}(t, t^0, v^0) \leq \phi_{(17)}(t, t^0, v^0).$$

The solutions are all confined to $\mathbb{R}^n_+$ and due to the ISS assumption they must exist for all times. It follows that any asymptotic bound on solutions of (17) is also an asymptotic bound on solutions of (15).

Given an initial state $v^0$ define $r^0$ to be the smallest $r \geq 0$ such that $\sigma(r) \geq v^0$. This turns out to be $r^0 = \max_i \sigma_i^{-1}(v_i^0)$. In particular, we have

$$\phi_{(15)}(t, t^0, v^0) \leq \phi_{(17)}(t, t^0, \sigma(r^0)) \tag{18}$$

for all $t \geq t^0$. Similarly, for any $v$ we can find an $r \geq \overline{r}$ such that $\sigma(r) \geq v$ and $g(\sigma(r)) + \rho(\overline{r}) \leq g(\sigma(r)) + \rho(r) \ll 0$. Also we have $g(\sigma(\overline{r})) + \rho(\overline{r}) \leq 0$.

Claim: For all $v > \sigma(\overline{r})$, $g(v) + \rho(\overline{r}) \not\geq 0$.

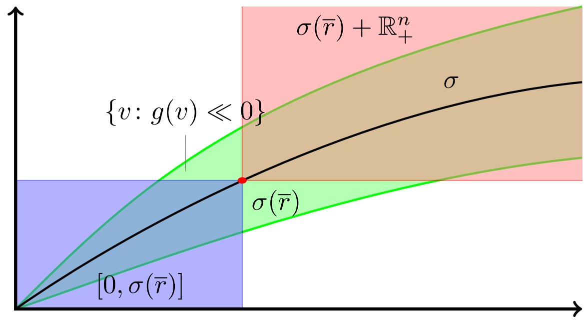


Fig. 1. The geometry underlying the proof of Theorem 4.8.

Proof. For any such $v$ there exists a minimal $r > \overline{r}$, such that $\sigma(r) \geq v$ and for some $i$, $\sigma_i(r) = v_i$. Since $g$ is quasi-monotone nondecreasing we have $g_i(v) + \rho(\overline{r}) \leq g_i(\sigma(r)) + \rho(\overline{r}) < 0$ by the definition of $\sigma$ and $\rho$. It follows that indeed $g(v) + \rho(\overline{r}) \not\geq 0$, proving the claim.

By Lemma 4.2 it follows that the set $[0, \sigma(\overline{r})]$ attracts all trajectories of (17) that start in $\sigma(\overline{r}) + \mathbb{R}^n_+$. By the ordering of solutions it follows that all trajectories that start in $\mathbb{R}^n_+$ are attracted to $[0, \sigma(\overline{r})]$. By an argument similar to that underling the stability assertion in Lemma 3.1, the set $[0, \sigma(\overline{r})]$ is also Lyapunov stable, hence globally asymptotically stable with respect to the system (17).

In other words, for system (15), inputs $w(\cdot)$ such that for almost all times $w(t) \in [0, \rho(\overline{r})]$, asymptotically yield states in $[0, \sigma(\overline{r})]$. This concludes the proof. ∎

## V. Numerical implementation

As seen in Theorem 4.8, the computation of asymptotic gain amounts to the computation of the path $\sigma(r)$, which is the trajectory of the autonomous comparison system (11) for a special initial condition $v^0$. In practice, even if we known that (10) is ISS, we generally do not know component-wise unbounded $\sigma$ and $\rho$ without any addition structure. However, on finite intervals we can find sampled versions of $\sigma$ and $\rho$ quite easily, allowing to generate plots of asymptotic gains of large-scale systems in an instant.

### A. Computing $\sigma$ and $\rho$

Assume we have an initial condition $s^R \in \mathbb{R}^n_+$ satisfying $g(s^R) \ll 0$, and, without loss of generality, $\|s^R\| = R$. Then we can numerically compute a solution $\phi(t, 0, s^R)$ of (2) for times $t \in [0, T]$ for some very large $T > 0$, yielding $\|\phi(T, 0, s^R)\| = \varepsilon > 0$ and $\varepsilon$ arbitrarily small. Due to Lemma 3.1.vi along this trajectory it must hold that $g(\phi(t, 0, s^R)) \ll 0$. Re-parametrisation of this trajectory gives a path $\sigma$ defined for $r \in [\varepsilon, R]$, normalized such that $\|\sigma(r)\|_1 = r$ for all $r \in [\varepsilon, R]$, and whose component functions $\sigma_i$ are strictly increasing and could be extended to $\mathcal{K}_\infty$ functions. The computation of $\rho$ remains the same as before, essentially it amounts to evaluation of the vector field $g$ along $\sigma$.

The conclusions of the main result remain essentially true with such a finite length path, but the sets of initial conditions and inputs have to be restricted, and the asymptotic gain will be biased.

*Corollary 5.1:* Let a large-scale system (6), decomposed into subsystems (5) satisfying (7)–(8), be given. Consider (10) with $g$ quasi-monotone nondecreasing and locally Lipschitz, $g(0) = 0$, and arising from (8). Let $\varepsilon, R \in \mathbb{R}_+$ satisfy $0 < \varepsilon < R$. Suppose that $\sigma, \rho \colon [\varepsilon, R] \to \mathbb{R}^n_+$ are positive, continuous, strictly increasing (in every component) such that $g(\sigma(r)) + \rho(r) \ll 0$ for all $r \in [\varepsilon, R]$. Then the gain of the magnitude of external disturbances to the asymptotic magnitude the states of the nominal system (6) is componentwise bounded from above by a map $\tilde{G} \colon \mathbb{R}_+ \to \mathbb{R}^n_+$ with $\tilde{G}_i \in \mathcal{K}_\infty$, given by

$$\tilde{G}(w) = \max\left\{G(w), \underline{\psi_1^{-1}} \circ \sigma(\varepsilon)\right\}$$

with $G$ and $\underline{\psi_1^{-1}}$ as in Theorem 4.8, in the sense that

$$\limsup_{t\to\infty} \underline{\|}x(t, x^0, u(t))\underline{\|}_{\mathcal{A}} \leq \tilde{G}(\underline{\|}u\underline{\|}_{L_\infty})$$

for all $L_\infty$ inputs $u$ such that a.e., $\underline{\|}u(t)\underline{\|} \in [0, \rho(R)]$ and all initial conditions $x^0 \in \mathbb{R}^N$ such that $\underline{\|}x^0\underline{\|}_{\mathcal{A}} \in [0, \sigma(R)]$.

The proof is essentially the same as that of Theorem 4.8. It has only to be noted that the set $[0, \sigma(R)]$ is invariant under the system $\dot{v} = g(v) + \rho(R)$, which ensures that solutions of the comparison system exist for all times.

Now the computation of the path only hinges upon knowing the initial condition $s^R \in \mathbb{R}^n_+$.

### B. Computing $s^R \in \mathbb{R}^n_+$

From the discussion in the previous section it remains to find the initial condition $s^R \in \mathbb{R}^n_+$ satisfying $\|s^R\|_1 = R$ and $g(s^R) \ll 0$.

If the origin is attractive with respect to (2) and the simplex set

$$S_R = \{v \in \mathbb{R}^n_+ \colon \|v\|_1 = R\}$$

is contained in the region of attraction, which we denote by $\mathcal{B}$, then Lemma 3.1 tells us that there exists such a point $s^R \in \mathbb{R}^n_+$.

Let $g$ be locally Lipschitz and quasi-monotone nondecreasing, $g(0) = 0$. Taking any initial condition $v^0 \in \mathbb{R}^n_+, v^0 \gg 0$ and computing $\phi(t, v^0)$ for $t \geq 0$ tells us whether or not the origin is attractive: If $\phi(t, v^0) \to 0$ as $t \to \infty$ then it is and $v^0 \in \mathcal{B}$, otherwise $v^0$ is at least not in the region of attraction. If $v^0 \in \mathcal{B}$, then also $[0, v^0] \subset \mathcal{B}$, and hence, with $R = \min_i v^0_i > 0$, $S_R \subset \mathcal{B}$.

At this stage, it must hold that for all $v \in S_R$, $g(v) \ngeq 0$. Now, using the labeling function

$$l_\delta(v) = \min\{i = 1, \ldots, n \colon g_i(v) + \delta < 0\}, \qquad (19)$$

where $\delta > 0$, as well as one of the two algorithms described in [8] yields the desired $s^R \in \mathbb{R}^n_+$, cf. also [6], [18].

*Remark 5.2:* Here $\delta > 0$ is a numerical design parameter. Larger $\delta$ give faster convergence of the Eaves algorithm, but, since not necessarily

$$g(v) + (\delta, \ldots, \delta)^T \ngeq 0 \text{ for all } v \in S_R, \qquad (20)$$

the algorithm may not converge to a point $s^R \in \mathbb{R}^n_+$ if $\delta$ is too large. On the other hand, if $g(v) \ngeq 0$ for all $v \in S_R$ then due to continuity of $g$, for small enough $\delta > 0$ it must converge to a point $s^R \in \mathbb{R}^n_+$, possibly at the cost of a lower rate of convergence.

### C. Sampling and interpolation of $\sigma$ and $\rho$

In practice the computation of $\sigma$ is achieved by numerical integration, first yielding a sampled version $\phi(t_k, s^R)$, $\{t_k\}_{k=0}^K$, of the trajectory $\phi(\cdot)$ in Section V-A. Reparametrization first yields $\{\sigma(r_k)\}_{k=0}^K$ with $\varepsilon = r_0 \leq r_1 \leq \ldots \leq r_K = R$, such that $\|\sigma(r_k)\|_1 = r_k$ for all $k$. To obtain $\rho$, one could fix $1 > \kappa > 0$, a design parameter which should be very small, and then compute directly

$$\rho(r_k) := \min_{l=k,\ldots,K} (1 - \kappa)(-g(\sigma(r_k))).$$

If the sampling intervals are small enough, then linear interpolation of the data points $\sigma(r_k)$ and $\rho(r_k)$ is permissible, yielding piecewise affine paths, with $\sigma(r) \in \Omega = \{v \in \mathbb{R}^n_+ \colon g(v) \ll 0\}$ and $g(\sigma(r)) + \rho(r) \ll 0$ for all $r \in [\varepsilon, R]$. This in particular makes it easy to compute the inverses $\rho_i^{-1} \colon [\rho_i(\varepsilon), \rho_i(R)] \to [\varepsilon, R]$, which are again piecewise affine.

### D. A few remarks

*Remark 5.3:* If the origin is not attractive with respect to (2) then in particular (10) is not ISS. If the origin is attractive with respect to (2) with a domain of attraction that contains, say $[0, v^0]$, $v^0 \gg 0$, then this does not imply that (10) is ISS, it only implies ISS *locally*. In the case when the system (2) is not globally ISS, its asymptotic gain can be computed (i.e., finite) only for small magnitudes of the external input. Corollary 5.1 will still produce a possibly large region $[0, \sigma(R)]$, but the corresponding region $[0, \rho(R)]$ might become small. It may be worth mentioning that the ISS property held only locally does not guarantee integral ISS.

*Remark 5.4:* The approach put forward in this paper can be a useful tool when large-scale systems are designed or analysed. Here it serves as a first step before one even tries to derive analytical results. If the proposed algorithm (Corollary 5.1) fails to deduce ISS even with respect to very small regions of initial states and inputs then any analytical approach to prove ISS of the comparison system (9) must fail as well.

## VI. Numerical examples

In this section we provide two examples. First we provide a nontrivial example for the application of Theorem 4.8. Then we provide some statistics on the numerical implementation of one of the Eaves algorithms [8], which is needed to compute the initial condition in Section V-B. We also show plots of some exemplary gains that we obtain with our approach for a particular nonlinear example.

*Example 6.1:* Consider $g\colon \mathbb{R}^4_+ \to \mathbb{R}^4$ given by

$$g(v) = \begin{pmatrix} -v_1^6 + v_4^4 \\ -3v_2^2 + v_3^3 + v_4^2 \\ -4v_3 + v_1 \\ -2v_4^2 + v_2^2 + v_3^3 \end{pmatrix}.$$

Observe that $g$ is locally Lipschitz and quasi-monotone nondecreasing(its Jacobian is Metzler) with $g(0) = 0$.

Claim: The system

$$\dot{v} = g(v) + w, \quad u, v \in \mathbb{R}^4_+, \tag{21}$$

is ISS from $w$ to $v$.

Proof. In virtue of Lemma 4.6 all we have to show is that there exist $\sigma_i, \rho_i \in \mathcal{K}_\infty$ such that with $\sigma(r) = (\sigma_1(r), \ldots, \sigma_4(r))^T$ and $\rho(r) = (\rho_1(r), \ldots, \rho_4(r))^T$,

$$g(\sigma(r)) + \rho(r) \ll 0 \text{ for all } r > 0.$$

Now take

$$\sigma(r) = \begin{pmatrix} \sqrt[3]{10r} \\ \sqrt{3r} \\ \sqrt[3]{3r} \\ \sqrt{4r} \end{pmatrix}$$

which is component-wise of class $\mathcal{K}_\infty$. Next we compute

$$g(\sigma(r)) = \begin{pmatrix} -84r^2 \\ -2r \\ \left(\sqrt[3]{10} - 4\sqrt[3]{3}\right)\sqrt[3]{r} \\ -2r \end{pmatrix} =: -\tilde{\rho}(r)$$

and observe that also $\tilde{\rho}$ is of class $\mathcal{K}_\infty$ in every component. Defining $\rho = \frac{1}{2}\tilde{\rho}$ yields the desired inequality and guarantees that indeed system (21) is ISS due to Lemma 4.6.

In fact, since $\tilde{\rho}$ already is strictly increasing, we can compute the asymptotic gain with respect to the max-norm as

$$\gamma(r) = \max_{i,j} \left(\sigma_i \circ \tilde{\rho}_j^{-1}\right)(r).$$

The gain has been plotted in Figure 2.

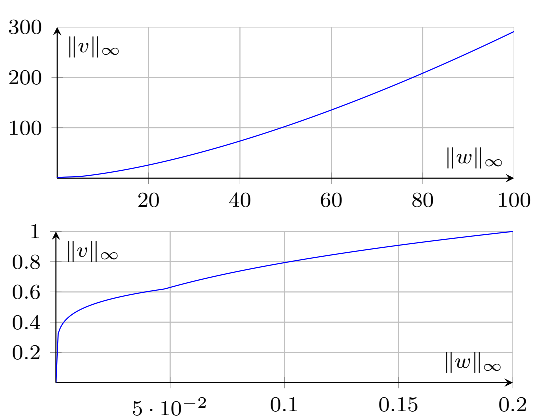


Fig. 2. The nonlinear gain obtained for the monotone system $\dot{v} = g(v) + w$ in Example 6.1 on two different intervals.

Since in general we cannot write down explicitly $\sigma$ and $\rho$, we have to resort to numerical integration to find $\sigma$. From there we can compute $\rho$ easily as was described in Section V. In the next example we use the Eaves algorithm [8] to compute the initial condition $s^R$ that is used in Section V for a number of randomly chosen monotone systems, to give an idea of the numerical complexity of this task. Note however, that here we use the K1 complex described in [8], which can be considered slow in comparison to similar algorithms.

*Example 6.2:* Assume that $A \in \mathbb{R}^{n\times n}$ is of the form $A = -I + P$, where $I$ is the identity and $P$ is a non-negative matrix (element-wise), with spectral radius $\rho(P) := \max\{|\lambda|\colon \lambda \text{ is an eigenvalue of } P\} < 1$. It can be verified that the spectral abscissa $\alpha(A)$ satisfies

$$\begin{aligned} \alpha(A) &:= \max\{\operatorname{Re}\lambda\colon \lambda \text{ is an eigenvalue of } A\} \\ &= -1 + \rho(P) < 0. \end{aligned}$$

So $A$ is a Hurwitz matrix with negative diagonal entries and non-negative off-diagonal entries. Now we define a nonlinear but smooth and order preserving coordinate transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $S(0) = 0$ and $S(\mathbb{R}^n_+) = \mathbb{R}^n_+$. Here we have chosen $S$ to be diagonal and given by

$$S(v)_i = \begin{cases} e^{v_i}/e & \text{if } v_i > 1, \\ v_i & \text{if } v_i \in [-1, 1], \\ -e^{-v_i}/e & \text{if } v_i < -1. \end{cases}$$

Now define the differential equation

$$\dot{v} = g(v) = S'(S^{-1}(v))AS^{-1}(v). \tag{22}$$

Under a nonlinear change of coordinates (22) is just the system

$$\dot{z} = Az, \tag{23}$$

and by our initial remark this system is stable.

Instead of computing the stable manifold for (23) and transforming it to the coordinates of system (22), we apply the Eaves algorithm to find points $s^R \in \mathbb{R}^n_+$ near the stable manifold of (22) satisfying $g(s^R) \ll 0$.

The Eaves-algorithm has been implemented (in MATLAB) as it is proposed in the paper [8] based on the K1 complex and using the integer labeling

$$l_\delta(v) = \min\{i\colon g_i(v) < -\delta\}, \tag{24}$$

where $\delta$ is numerical design parameter and usually chosen very small. It should be noted that larger $\delta$ give faster convergence, while for $\delta \to 0$ convergence is guaranteed, cf. Table I.

The algorithm is then applied to the simplex $S_R = \{v \in \mathbb{R}^n_+\colon \|v\|_1 = R\}$ and produces $s^R \in \mathbb{R}^n_+$.

As a proof of concept, numerical simulations have been performed on a MacBook with 2GB RAM and Intel Core 2 Duo Processor operating at 2.4 GHz in MATLAB under MacOS. The outcomes are shown in Table I, with average run times and number of iterations of the Eaves-Algorithm for a range of dimensions $n$ of the state space.

| Dimension ($n$) | Avg.Time [s] | Avg.Iterations | Succ.rate |
|---|---|---|---|
| $\delta = 0.1$, Max.Iterations=100,000 | | | |
| 5 | 0.10610 | 340.48 | 100% |
| 10 | 0.64374 | 2063.87 | 100% |
| 15 | 1.77744 | 5549.80 | 100% |
| 20 | 3.74164 | 11534.94 | 100% |
| 25 | 7.57742 | 20994.42 | 100% |
| 30 | 13.80078 | 36864.52 | 99% |
| $\delta = 0.5$, Max.Iterations=100,000 | | | |
| 5 | 0.02906 | 91.83 | 94% |
| 7 | 0.06723 | 215.47 | 92% |
| 10 | 0.16074 | 510.90 | 10% |
| $\delta = 0.01$, Max.Iterations=100,000 | | | |
| 5 | 0.64410 | 2131.84 | 100% |
| 7 | 2.00263 | 6500.52 | 100% |
| 10 | 5.06990 | 16191.96 | 100% |

TABLE I

The Eaves K1 algorithm applied to the example with $A \in \mathbb{R}_+^{n\times n}$, $A = -I + P$, $R = 10$, $\alpha(A) = -0.2$, and different values of $\delta > 0$ (the labeling parameter). Here the matrix $P$ described above is populated with uniformly distributed positive random entries, then 30% of these are set to zero. The results shown are time in seconds and number of iterations needed per *successful* run of the algorithm, as well as the success rate. The numbers are averages over 100 simulations. The effect of different choices of $\delta$ is evident.

## VII. Conclusions

We have demonstrated that comparison principles and vector Lyapunov functions lead to simple formulae for asymptotic gains by utilizing one special path in an invariant set of the autonomous dynamics of the comparison system. When this approach is combined with a numerical implementation of the Eaves algorithm it also provides a useful tool to check whether a large-scale interconnection is IS(p)S. For such an implementation it has been shown that comparison systems of order up to 30 (i.e., 30 interconnected systems) are numerically feasible. Using similar arguments, it is possible to also derive transient bounds in a similar way.

## References


[1] D. Angeli and A. Astolfi, "A tight small-gain theorem for not necessarily ISS systems," *Systems Control Lett.*, vol. 56, no. 1, pp. 87–91, 2007.

[2] D. Angeli and E. D. Sontag, "Monotone control systems," *IEEE Trans. Autom. Control*, vol. 48, no. 10, pp. 1684–1698, 2003.

[3] D. Angeli, E. D. Sontag, and Y. Wang, "A characterization of integral input-to-state stability," *IEEE Trans. Autom. Control*, vol. 45, no. 6, pp. 1082–1097, 2000.

[4] J.-P. Aubin, *Viability theory*, ser. Systems & Control: Foundations & Applications. Boston, MA: Birkhäuser, 1991.

[5] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth, "An ISS small-gain theorem for general networks," *Math. Control Signals Syst.*, vol. 19, no. 2, pp. 93–122, May 2007.

[6] ——, "Numerical verification of local input-to-state stability for large networks," in *Proc. 46th IEEE Conf. Decis. Control*, New Orleans, LA, USA, 2007, pp. 4471–4476.

[7] ——, "Small gain theorems for large scale systems and construction of ISS Lyapunov functions," *SIAM J. Control Optim.*, 2010, *To appear, accepted* March 25, 2010, arXiv:0901.1842.

[8] B. C. Eaves, "Homotopies for computation of fixed points," *Math. Programming*, vol. 3, pp. 1–22, 1972.

[9] G. Enciso and E. D. Sontag, "Monotone systems under positive feedback: multistability and a reduction theorem," *Systems Control Lett.*, vol. 54, no. 2, pp. 159–168, 2005.

[10] S. Huang, M. R. James, D. Nešić, and P. M. Dower, "Analysis of input-to-state stability for discrete time nonlinear systems via dynamic programming," *Automatica J. IFAC*, vol. 41, no. 12, pp. 2055–2065, 2005.

[11] Z.-P. Jiang, A. R. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Math. Control Signals Systems*, vol. 7, no. 2, pp. 95–120, 1994.

[12] I. Karafyllis and Z.-P. Jiang, "A vector small-gain theorem for general nonlinear control systems," *Submitted to IEEE Trans. Automat. Control*, 2009, arXiv:0904.0755v1.

[13] ——, "A vector small-gain theorem for general nonlinear control systems," in *Proc. Joint 48th IEEE Conf. Decis. Control and 28th Chinese Contr. Conf.*, Shanghai, P.R.China, December 2009, pp. 7996–8001.

[14] V. Lakshmikantham and S. Leela, *Differential and integral inequalities: Theory and applications. Vol. I: Ordinary differential equations*. New York: Academic Press, 1969.

[15] T. Liu, D. J. Hill, and Z.-P. Jiang, "Lyapunov formulation of ISS small-gain in dynamical networks," in *Proc. Joint 48th IEEE Conf. Decis. Control and 28th Chinese Contr. Conf.*, Shanghai, P.R.China, 2009, pp. 4204–4209.

[16] M. Malisoff and P. De Leenheer, "A small-gain theorem for monotone systems with multivalued input-state characteristics," *IEEE Trans. Autom. Control*, vol. 51, no. 2, pp. 287–292, 2006.

[17] D. Raimondo, L. Magni, and R. Scattolini, "Decentralized MPC of nonlinear systems: An input-to-state stability approach," *International Journal of Robust and Nonlinear Control*, vol. 17, no. 17, pp. 1651–1667, 2007.

[18] B. S. Rüffer, P. M. Dower, and H. Ito, "Applicable comparison principles in large-scale system analysis," in *Proc. of the 10th SICE Annual Conference on Control Systems*, Kumamoto, Japan, March 2010.

[19] B. S. Rüffer, C. M. Kellett, and S. R. Weller, "Connection between cooperative positive systems and integral input-to-state stability of large-scale systems," *Automatica J. IFAC*, 2010, *To appear, accepted* January 26, 2010.

[20] H. L. Smith, *Monotone dynamical systems*, ser. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 1995, vol. 41.

[21] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Autom. Control*, vol. 34, no. 4, pp. 435–443, 1989.

[22] E. D. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE Trans. Autom. Control*, vol. 41, no. 9, pp. 1283–1294, 1996.


## Appendix

*Proof of Lemma 4.2* We start with the second assertion. Suppose it does not hold. Then there exists an $\hat{x} > \overline{x}$ with $f(\hat{x}) \geq 0$. If $f(\hat{x}) = 0$ then we are done, $\hat{x}$ an equilibrium and the corresponding solution not attracted to $\overline{x} - \mathbb{R}^n$, contradicting the assumption.

It suffices to show that for all $\tilde{x} > \hat{x}$, $\tilde{x} \not\gg x$ (i.e., those $\tilde{x}$ on the boundary of $\hat{x} + \mathbb{R}^n$) we have $f(\tilde{x}) \geq 0$. This follows directly from the fact that $f$ is quasi-monotone nondecreasing. Under a change of coordinates $x = \hat{x} + z$, the origin becomes the point of interest; let $\tilde{f}(z) = f(\hat{x} + z)$. Then $\dot{x} = f(x) = \tilde{f}(z) = \dot{z}$, and $\tilde{f}(0) \geq 0$. Now by a standard viability theory argument (cf. [4], [19, Lemma 3.3]), the positive orthant is invariant for system $\dot{z} = \tilde{f}(z)$, hence the set $\hat{x} + \mathbb{R}^n$ is invariant for system $\dot{x} = f(x)$. But this contradicts attractivity of the set $\overline{x} - \mathbb{R}^n$, proving the second assertion.

Now the first assertion. Consider again the system $\dot{z} = \tilde{f}(z)$ instead with $z = x - \overline{x}$. Then we have to show that $-\mathbb{R}^n_+$ is globally attractive.

By assumption $\tilde{f}(z) \not\geq 0$ for all $z > 0$ and $\tilde{f}(0) \leq 0$. It follows by the same argument as in Lemma 3.1.v that the set

$$\tilde{\Omega} := \{z \in \mathbb{R}^n \colon \tilde{f}(z) \ll 0\}$$

satisfies

$$\tilde{\Omega} \cap S_r \neq \emptyset \text{ for all } r > 0.$$

By [19, Lemma 3.13] the set $\tilde{\Omega}$ is forward invariant. It follows that for any $z \in \tilde{\Omega} \cap \mathbb{R}^n_+$, $\phi(t,z) \to -\mathbb{R}^n_+$.

Now, by assumption for any $z \in \mathbb{R}^n_+$ there exists $y \in \mathbb{R}^n_+$, $y \geq z$, such that $\tilde{f}(y) \ll 0$. By the ordering of solutions property of monotone flows (Lemma 3.1, possibly applied several times in succession), $\phi(t,z) \leq \phi(t,y) \to -\mathbb{R}^n_+$. This proves that indeed $-\mathbb{R}^n_+$ is globally attractive for system $\dot{z} = \tilde{f}(z)$, and hence $\overline{x} - \mathbb{R}^n_+$ is globally attractive for the system $\dot{x} = f(x)$. ∎

*Proof of Lemma 4.3* The first part of the result is obvious. For the proof of the second part assume first that system (10) is ISS.

This implies that for any constant input $w$ there exists a bounded set $A_w \subset \mathbb{R}^n_+$ such that $A_w$ is globally asymptotically stable with respect to the monotone system $\dot{v} = g_w(v)$ with $g_w := g(v) + w$. Denote $v_w := \sup A_w$, so that $[0, v_w] \supset A_w$. Due to the monotonicity of the flow, also $[0, v_w]$ is globally attractive. By Lemma 4.2 it follows that $g_w(v) \not\geq 0$ for all $v > v_w$. Further, as in the proof of said lemma, it follows that we have

$$\{v \colon v \geq v_w\} \cap \{v \colon g_w(v) \ll 0\} \cap S_r \neq \emptyset$$

for all $r > \|v_w\|_1$. In particular there exists $v > v_w$ such that $g(v) + w = g_w(v) \ll 0$. Hence, for any $w \in \mathbb{R}^n_+$ there exists a $v \in \mathbb{R}^n_+$ such that $g(v) + w \ll 0$, or, in other words,

$$-g(v) \gg w. \tag{25}$$

Now we have to show that also for any $\overline{v}$ and $w$ there exists a $v > \overline{v}$ such that $g(v) + w \ll 0$. Given $\overline{v}$ and $w$, let $\tilde{v} := \max\{\overline{v}, v_w\}$ with $v_w := \max A_w$ as before. Let $\tilde{w} := \max_{v \in [0,\tilde{v}]} |g(v)|$. Then we have $g(\tilde{v}) + \tilde{w} \leq 0$ and the set $[0, \tilde{v}]$ is globally attractive with respect to $\dot{v} = g_{\tilde{w}}$.

Repeating the argument above, we obtain a point $v > \tilde{v} \geq \overline{v}$ such that $g(v) + w \leq g(v) + \tilde{w} \ll 0$. ∎

*Proof of Lemma 4.4* The first part follows from Lemma 3.1.vii. For the second part we note that closedness of $\mathcal{O}$ is obvious. Now let $\xi : (0, \infty) \to \mathbb{R}$ be any strictly decreasing diffeomorphism (in contrast to strictly increasing). Then it is plain to check that

$$\tilde{\sigma}(s) := \begin{cases} \tilde{\phi}(\xi(s)) & \text{if } s > 0 \\ 0 & \text{otherwise} \end{cases}$$

defines such a path, except that not necessarily $\|\tilde{\sigma}(r)\|_1 = r$. Normalization yields $\sigma(r) = \frac{1}{\|\tilde{\sigma}(r)\|_1}\tilde{\sigma}(r)$. The third part is a consequence of the fact that $g(\tilde{\phi}(t)) \ll 0$. It suffices to consider $\tilde{\sigma}$ instead of $\sigma$, which satisfies $\frac{d}{dr}\tilde{\sigma}(r) = \underbrace{g_i(\tilde{\phi}(\xi(r)))}_{<0} \cdot \underbrace{\xi'(r)}_{<0} > 0$ for all $r > 0$. Hence already the functions $\tilde{\sigma}_i$ are of class $\mathcal{K}$ and normalization does not change this fact. Since the solution $\tilde{\phi}$ exists for all times and is strictly increasing in negative time, it must either tend to an equilibrium point, which it does not, or be unbounded. Hence at least one $\sigma_i$ must be unbounded. ∎

*Proof of Lemma 4.5* The fact that $-g(\sigma(r)) \gg 0$ for all $r > 0$ is obvious. Any unbounded, positive definite function $\mathbb{R}_+ \to \mathbb{R}_+$ can be bounded from below by a positive definite, continuous, nondecreasing, and unbounded function, e.g.,

$$\overline{\rho}_i(r) = \inf_{s \geq r} \tilde{\rho}_i(s)$$

is suitable. This in turn can be bounded from below by a class $\mathcal{K}_\infty$ function, using a standard integral argument. ∎

*Proof of Lemma 4.6* This follows from Lemma 4.3.ii using the fact that for any $v, w \in \mathbb{R}^n_+$ there exists an $r > 0$ such that $\sigma(r) \geq v$ and $\rho(r) \geq w$. ∎