CONVERGENCE PROPERTIES FOR DISCRETE-TIME NONLINEAR SYSTEMS

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ABSTRACT. Three similar convergence notions are considered. Two of them are the long established notions of convergent dynamics and incremental stability. The other is the more recent notion of contraction analysis. All three convergence notions require that all solutions of a system converge to each other. In this paper, we investigate the differences between these convergence properties for discrete-time, time-varying nonlinear systems by comparing the properties in pairs and using examples. We also demonstrate a time-varying smooth Lyapunov function characterization for each of these convergence notions. In addition, with appropriate assumptions, we provide several sufficient conditions to establish relationships between these properties in terms of Lyapunov functions.

1. INTRODUCTION

Convergence properties, namely *incremental stability* [1–3], *convergent dynamics* [4–6], and *contraction analysis* [7], are established methods to characterize the asymptotic behavior of one solution with respect to any other solution of a (nonlinear) dynamical system. In particular, all convergence properties impose conditions that all solutions "forget" their initial conditions and converge to each other. This is a highly desirable property in solving problems in nonlinear output stabilization and regulation [3, 5], synchronization [2, 8–10], observer design [2, 7], steady-state and frequency response analysis of nonlinear systems [11–13], and others. These three notions of convergence are similar and largely related to each other. However, the three methods were all derived independently, were motivated distinctly, and employ different tool sets. As a consequence, their mutual relationships are, in general, not fully understood even for continuous-time nonlinear systems in the literature. Thus, the purpose of this study is to investigate explicit differences between these properties.

Incremental stability describes the asymptotic property of differences between any two solutions. Specifically, an augmented system is formed by two "copies" of the original system. Then, global asymptotic stability of a special closed set (called the diagonal) with respect to the augmented system can be demonstrated to be equivalent to incremental stability of the original system. Thus, a Lyapunov characterization for incremental stability of the original system can be derived from a classical Lyapunov characterization of global asymptotic stability. Historically, the origin of

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incremental stability goes back to works on forced oscillators by German mathematicians Trefftz [14] and Reissig [15–17], without the property being named. It was LaSalle who in [18] attributes the term "extreme stability" to Reissig's work [15]. Subsequently, the work of Yoshizawa [1, Chapter IV] made significant contribution to the extreme stability area. In particular, for continuous-time, time-varying systems, a continuous Lyapunov function characterization of extreme stability was demonstrated [1, Theorem 21.1]. Similar results are found in [2, Theorem 1] and [19, Theorem 5] for time-invariant and time-varying systems, respectively, where in both works the property is termed incremental stability. Incremental Input-to-State Stability [2] is an extension of incremental stability to systems with input using the Input-to-State Stability (ISS) framework [20].

Convergent dynamics, on the other hand, requires the existence of a unique and asymptotically stable reference solution that is bounded for all (backward and forward) times. However, a priori knowledge of this reference solution is not necessarily required. Every other solution, then, converges to this reference solution asymptotically. A convenient, sometimes easy to check, sufficient condition of convergent dynamics is the Demidovich condition. Essentially, using a quadratic Lyapunov function, exponential convergence of all solutions to a reference solution can be demonstrated (see, for instance, [6]). The existence and boundedness of the reference solution usually relies on Yakubovich's Lemma; i.e., that a compact and positively invariant set contains at least one bounded solution defined for all times. This result was demonstrated in [21, Lemma 2] based on ideas of Demidovich in [22, 23]. Convergent dynamics (or convergent systems) was pioneered by Demidovich [4] (only available in Russian), see also a historical perspective of developments for convergent dynamics in [6]. A converse Lyapunov theorem for globally convergent systems was introduced in [19, Theorem 7]. More details and further extensions of convergent dynamics to systems with input (input-to-state convergence) and applications to output regulation problems can be found in [5].

Lastly, contraction analysis [7] was inspired by fluid mechanics and differential geometry. Contraction analysis utilizes local analysis of the linearization along every trajectory to establish globally convergent behavior. This method, in essence, generalizes linear eigenvalue analysis to nonlinear systems. As a result, contraction analysis provides a characterization for exponential convergence of one solution with respect to any other solution. This approach, thus, independently extends the Demidovich sufficient condition to a necessary and sufficient condition. Contraction analysis (or contraction metrics) was first introduced in [7]. Since then, developments based on differential geometry have been made in [24], [25]. A historical perspective of contraction analysis, including earlier closely related concepts, is presented in [26]. Various applications of contraction analysis can be found in [8, 27].

In the continuous-time setting, several relationships have been established between these properties. Available comparisons include convergent dynamics versus incremental stability [19] and extending contraction analysis and incremental stability to the same differential geometric framework [24].

In this paper, we study and compare the three previously discussed convergence properties for time-varying nonlinear systems, albeit, in contrast to most of the existing literature, in discrete time. In particular, we show that convergent dynamics and incremental stability are two distinct properties. Similarly, convergent dynamics and contraction analysis are two distinct properties. However, contraction analysis is a special case of incremental stability, namely, exponential incremental stability. Furthermore, we establish various sufficient conditions to demonstrate relationships between the three convergence properties. This is done by either tightening the regularity requirements on the dynamics or by adding an assumption on the state space. We also provide time-dependent smooth Lyapunov function characterizations for each of the properties. Furthermore, we present discrete-time analogues of the Demidovich condition for incremental stability (similar to the continuous-time result [19, Equation 15]), convergent dynamics (similar to the continuous-time result [6, Theorem 1]), and contraction analysis.

To derive the three Lyapunov function characterizations, we rely heavily on a converse Lyapunov result for time-varying asymptotically stable systems [28, Theorem 1] and a stronger version of this theorem for time-varying exponentially stable systems [28, Theorem 2]. Specifically, we use a result [2, Lemma 2.3] to convert global asymptotic (or exponential) incremental stability to global asymptotic (or exponential) stability of a closed set with respect to an augmented system. For convergent dynamics, we first apply a nonlinear change of coordinates with respect to the reference solution, then we use a standard Lyapunov result [28, Theorem 1] (similar to [29, Theorem 23] for the continuous-time case) to obtain the desired characterization. Lastly, we demonstrate that contraction analysis, in fact, is equivalent to exponential incremental stability, and hence, is equivalent to the existence of an exponential incremental Lyapunov function.

To demonstrate the differences between the three considered convergence notions, we construct examples of systems that individually satisfy one of the three properties but not the others. Specifically, we exploit the fact that: global incremental stability (any two trajectories tend to each other asymptotically) is a stronger property than global asymptotic convergence to a single trajectory. From there, we are able to construct a system (Example1) showing that convergent dynamics does not imply incremental stability. On the other hand, we exploit another fact of convergent systems: that there must exist at least one bounded solution. This is not necessarily true for asymptotically incrementally stable systems. Thus, we are able show that incremental stability does not imply convergent dynamics (Example 2).

In contrast to the other two properties, contraction analysis explicitly requires continuously differentiable dynamics. Given a system with differentiable right hand side, we are able to show that contraction analysis is equivalent to exponential incremental stability (Theorem 14). As a consequence, contraction analysis does not imply convergent dynamics. Finally, we employ the fact that the convergence rate in contraction analysis is always exponential. This is not necessarily true for convergent systems (Example 3). Therefore, contraction analysis and convergent dynamics are distinct convergence notions.

The paper is organized as follows: the necessary technical assumptions, notational conventions, and definitions of convergence properties are provided in Section 2. Then, Lyapunov function characterizations for these notions are presented in Section 3. In Section 4, comparisons in pairs for the three properties are presented together with various examples and sufficient conditions. Conclusions and future research indications are provided in Section 5 and several proofs are collected in the appendix.

2. PRELIMINARIES

We consider discrete-time nonlinear time-varying systems described by the difference equation

$$x(k+1) = f(k, x(k)), \quad x(k) \in \mathbb{R}^n, k \in \mathbb{Z}, \tag{1}$$

where $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $x(k_0) = \xi \in \mathbb{R}^n$ for $k_0 \in \mathbb{Z}$. For any $k_0 \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$, the semi flow of system (1) is a function $\phi : \mathbb{Z}_{\geq k_0} \to \mathbb{R}^n$, parameterized by initial state and time, satisfying $\phi(k_0; k_0, \xi) = \xi$ and equation (1); i.e., $\phi(k+1; k_0, \xi) = f(k, \phi(k; k_0, \xi))$ for all $k, k_0 \in \mathbb{Z}$ such that $k \geq k_0$. We use standard comparison function classes[1] $\mathcal{K}$, $\mathcal{L}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ (see [30]). For a vector $y \in \mathbb{R}^n$, a matrix $P \in \mathbb{R}^{n \times n}$, and a positive definite matrix Q we denote the Euclidean norm by $|y|$, the induced (matrix) norm by $\|P\| = \max_{x \in \mathbb{R}^n,\ x \neq 0} \dfrac{|Px|}{|x|}$, and the induced norm of y with respect to matrix Q by $\|y\|_Q = \sqrt{y^T Q y}$. Given two symmetric matrices A and B in $\mathbb{R}^{n \times n}$ we write $A \preceq B$ if for all $x \in \mathbb{R}^n$, $x^T A x \leq x^T B x$. We say that a set $\mathbb{X} \subseteq \mathbb{R}^n$ is positively invariant under (1) if for all $\xi \in \mathbb{X}$ and $k \geq k_0$, we have $\phi(k; k_0, \xi) \in \mathbb{X}$.

In the following subsections, we recall the definitions of incremental stability, convergent dynamics, and contraction analysis.

2.1. Incremental Stability.

The following definition of asymptotic incremental stability is a discrete-time analogue to that in [2, Definition 2.1].

Definition 1. System (1) is *uniformly asymptotically incrementally stable* in a positively invariant set $\mathbb{X} \subseteq \mathbb{R}^n$ if there exists $\beta \in \mathcal{KL}$ such that

$$|\phi(k; k_0, \xi_1) - \phi(k; k_0, \xi_2)| \leq \beta(|\xi_1 - \xi_2|, k - k_0), \tag{2}$$

holds for all $\xi_1, \xi_2 \in \mathbb{X}$ and $k \geq k_0$. In case $\mathbb{X} = \mathbb{R}^n$, we say that system (1) is *uniformly globally asymptotically incrementally stable.*

A strictly stronger property requires an exponential rate of convergence.

Definition 2. System (1) is *uniformly exponentially incrementally stable* in a positively invariant set $\mathbb{X} \subseteq \mathbb{R}^n$ if there exist $\kappa \geq 1$ and $\lambda > 1$ such that

$$|\phi(k; k_0, \xi_1) - \phi(k; k_0, \xi_2)| \leq \kappa |\xi_1 - \xi_2| \lambda^{-(k-k_0)} \tag{3}$$

[1] Recall that $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. If $\alpha \in \mathcal{K}$ is unbounded, it is of class-$\mathcal{K}_\infty$. A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{L}$ if it is continuous, strictly decreasing, and $\lim_{t \to \infty} \sigma(t) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if it is class-$\mathcal{K}$ in its first argument and class-$\mathcal{L}$ in its second argument. By convention, $\beta \in \mathcal{KL}$ satisfies $\beta(0, t) = 0$ for all $t \in \mathbb{R}_{\geq 0}$.

holds for all $\xi_1, \xi_2 \in \mathbb{X}$ and $k \geq k_0$. In case $\mathbb{X} = \mathbb{R}^n$, we say that system (1) is *uniformly globally exponentially incrementally stable.*

2.2. **Convergent Dynamics.** The following definition of convergent dynamics for discrete-time systems is recalled from [11, Definition 1]. This definition is a discrete-time analogue to the continuous-time definition in [6, Definition 1].

Definition 3. System (1) is *uniformly convergent* in a positively invariant set $\mathbb{X} \subseteq \mathbb{R}^n$ if

(1) there exists a unique solution $\bar{x}(k)$ of system (1) defined in $\mathbb{X}$ and bounded for all $k \in \mathbb{Z}$;
(2) there exists a function $\beta \in \mathcal{KL}$ such that, for all $\xi \in \mathbb{X}$ and $k \geq k_0$, we have

$$|\phi(k; k_0, \xi) - \bar{x}(k)| \leq \beta(|\xi - \bar{x}(k_0)|, k - k_0). \tag{4}$$

In case $\mathbb{X} = \mathbb{R}^n$, we say that system (1) is *uniformly globally convergent.*

Remark 4. In continuous-time, the definition of convergent dynamics [6, Definition 1] requires an additional condition that system (1) is forward complete. In discrete-time, such a condition is not required.

Remark 5. In fact, condition 2) and the boundedness of $\bar{x}(k)$ in Definition 3 also imply that $\bar{x}(k)$ is unique. Indeed, let $\hat{x}(k)$ be another solution defined and bounded for all $k \in \mathbb{Z}$, then $|\hat{x}(k_0) - \bar{x}(k_0)|$ is bounded for all $k_0 \in \mathbb{Z}$. Since (4) holds for all solutions, taking the limit for $k_0 \to -\infty$ in (4), it follows that $|\hat{x}(k) - \bar{x}(k)| \leq 0$ for all $k \in \mathbb{Z}$. Hence, $\hat{x} \equiv \bar{x}$.

2.3. **Contraction Analysis.** Unlike the previous two properties, contraction analysis explicitly requires the dynamics of the considered system to be continuously differentiable. As a consequence, in this subsection, we make a standing assumption that the mapping f (or the right hand side) of system (1) is continuously differentiable in x on $\mathbb{R}^n$.

We are now ready to state the formal definition (following [7]) of contraction analysis.

Definition 6. Suppose the right hand side of system (1) is continuously differentiable in x on a positively invariant set $\mathbb{X} \subseteq \mathbb{R}^n$. Then system (1) is uniformly contracting in $\mathbb{X}$ if there exist a nonsingular-matrix-valued function $\Theta : \mathbb{Z} \times \mathbb{X} \to \mathbb{R}^{n \times n}$ and constants $\mu, \eta, \rho > 0$ such that, for all $x \in \mathbb{X}, k \in \mathbb{Z}$, we have

$$\eta I \preceq \Theta(k,x)^T \Theta(k,x) \preceq \rho I, \tag{5}$$

$$F(k,x)^T F(k,x) - I \preceq -\mu I, \tag{6}$$

where the matrix $F(k,x)$ is given by

$$F(k,x) = \Theta(k+1,x) \frac{\partial f}{\partial x}(k,x) \Theta(k,x)^{-1}. \tag{7}$$

In case $\mathbb{X} = \mathbb{R}^n$, we say that system (1) is *uniformly globally contracting.*

Remark 7. Note that [7, Definition 3] includes (6) but not (5). However, at least the lower bound in (5) is required to guarantee contraction as the following example illustrates. The upper bound guarantees uniformity of the contraction property with respect to initial time, which is generally desirable. Consider the system

$$x(k+1) = (k^2+1)x(k), \quad x(k) \in \mathbb{R}^n, k \in \mathbb{Z}. \tag{8}$$

Choose $\Theta(k,x) := \frac{1}{k^2+1}$, then $F(k,x) = \frac{1}{k^2+2k+2}$ and, thus, $F(k,x)^T F(k,x) = \frac{1}{(k^2+2k+2)^2} < \frac{1}{4}$; i.e., condition (6) is satisfied. However, this system is unstable by inspection. Note that for this particular choice of $\Theta(k,x)$, the lower bound of condition (5) is not satisfied.

Given (1), similar to [24], we define the variational system

$$\begin{cases} x(k+1) = f(k, x(k)), \\ x_\delta(k+1) = \dfrac{\partial f}{\partial x}(k, x(k)) x_\delta(k). \end{cases}$$

As in [7], we refer to (9b) as the displacement dynamics where x_δ is called a displacement of x.

3. Lyapunov function characterizations

In this section, we present time-varying Lyapunov function characterizations for the three considered convergence properties.

3.1. Incremental Stability.

A Lyapunov function characterization of incremental stability for continuous-time systems was first presented in [1, Theorem 21.1 & Theorem 21.2]. Subsequently, similar results for time-invariant and time-varying systems were given in [2, Theorem 1] and [19, Theorem 5], respectively. The following theorem is a discrete-time analogue to [19, Theorem 5].

Theorem 8. *System* (1) *is uniformly globally asymptotically incrementally stable if and only if there exist a smooth function $V : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that*

$$\alpha_1(|x_1 - x_2|) \leq V(k, x_1, x_2) \leq \alpha_2(|x_1 - x_2|), \tag{10}$$

$$V\Big(k+1, f(k,x_1), f(k,x_2)\Big) - V(k, x_1, x_2) \leq -\alpha_3(|x_1 - x_2|) \tag{11}$$

hold for all $x_1, x_2 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$.

The proof of Theorem 8 is contained in the appendix. A function V satisfying (10)–(11) is called an incremental stability Lyapunov function.

3.2. Convergent Dynamics.

The following theorem is a time-varying Lyapunov function characterization of discrete-time globally convergent systems. This result is analogous to a continuous-time result presented in [19, Theorem 7].

Theorem 9. *Assume that system* (1) *is uniformly globally convergent. Then, there exist a smooth function* $V : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, *a constant* $c \geq 0$, *and functions* $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ *such that*

$$\alpha_1(|x - \bar{x}(k)|) \leq V(k,x) \leq \alpha_2(|x - \bar{x}(k)|), \tag{12}$$

$$V\Big(k+1, f(k,x)\Big) - \ V(k,x) \leq -\alpha_3(|x - \bar{x}(k)|), \tag{13}$$

$$V(k,0) \leq c < +\infty \tag{14}$$

hold for all $x \in \mathbb{R}^n$ *and* $k \in \mathbb{Z}$. *Conversely, if a smooth function* $V : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, *a constant* $c \geq 0$, *and functions* $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ *are given such that for some trajectory* $\bar{x} : \mathbb{Z} \to \mathbb{R}^n$, *estimates* (12)–(14) *hold, then system* (1) *is uniformly globally convergent and the solution* $\bar{x}(k)$ *is the unique bounded solution as in Definition 3.*

The proof of Theorem 9 is contained in the appendix. A function V satisfying (12)–(14) is called a convergent dynamics Lyapunov function.

3.3. **Contraction Analysis.** The following theorem is a Lyapunov function characterization for globally contracting systems. Note that this result is a special case of the Lyapunov function characterization for uniform global asymptotic incremental stability.

Theorem 10. *Assume the right hand side* f *of* (1) *is continuously differentiable in* x. *System* (1) *is globally contracting if and only if there exist a continuous function* $V : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, *and constants* $c_1, c_2, c_3 \in \mathbb{R}_{>0}$ *such that*

$$c_1|x_1 - x_2|^2 \leq V(k, x_1, x_2) \leq c_2|x_1 - x_2|^2, \tag{15}$$

$$\begin{aligned} &V\Big(k+1, f(k,x_1), f(k,x_2)\Big) - \ V(k,x_1,x_2) \\ &\quad \leq -c_3|x_1 - x_2|^2 \end{aligned} \tag{16}$$

hold for all $x_1, x_2 \in \mathbb{R}^n$ *and* $k \in \mathbb{Z}$.

The proof of Theorem 10 is contained in the appendix. A function V satisfying (15)–(16) is called a contraction analysis Lyapunov function.

4. Comparisons

In this section, we compare the three previously discussed properties in pairs. We provide several examples of systems that highlight the essential differences among the three convergence notions. Several sufficient conditions are also proposed to establish mutual relationships. Finally, we provide a Demidovich-like condition that is sufficient for all three properties.

4.1. **Incremental Stability vs. Convergent Dynamics.** The following example, which is adapted from [19, Example 3], provides a system that is uniformly globally convergent but not uniformly globally asymptotically incrementally stable.

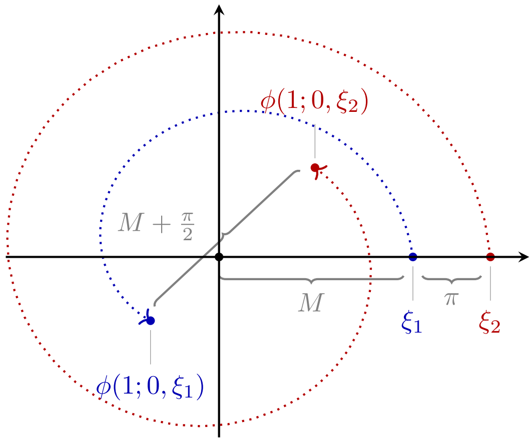

FIGURE 1. The two trajectories of the system defined in Example 1 start on the positive real half line with an initial separation of π at time $k=0$ and the initial distances to the origin are M and $M+\pi$. At time $k=1$, the polar arguments of the trajectories are shifted by 180° so that the separation distance is $M+\frac{\pi}{2}$.

Example 1. Consider the following system

$$x(k+1) = f(x(k)) := \frac{1}{2}\begin{bmatrix}\cos(|x(k)|) & -\sin(|x(k)|)\\ \sin(|x(k)|) & \cos(|x(k)|)\end{bmatrix} x(k) \tag{17}$$

where $x \in \mathbb{R}^2$. Consider the equilibrium $\bar{x}(k) = 0$ for all $k \in \mathbb{Z}_{\geq 0}$. Clearly, $\bar{x}(k)$ is bounded for all $k \in \mathbb{Z}$. Now, consider the quadratic Lyapunov function $V(x) = x^T x$. Then, it follows that

$$\begin{aligned} &V(f(x)) - V(x)\\ &\qquad = f(x)^T f(x) - x^T x\\ &\qquad = \frac{1}{4}x^T x - x^T x\\ &\qquad = -\frac{3}{4}x^T x \leq 0. \end{aligned}$$

Hence, appealing to Theorem 9, system (17) is uniformly globally convergent.

Converting system (17) to polar coordinates yields

$$\begin{aligned} r(k+1) &= \frac{r(k)}{2},\\ \theta(k+1) &= \theta(k) + r(k), \end{aligned}$$

whose explicit solution for an initial condition $(r_0, \theta_0) \in \mathbb{R} \times \mathbb{R}$ is given by

$$\begin{aligned} r(k) &= \frac{r_0}{2^k},\\ \theta(k) &= \theta_0 + r_0 \sum_{\kappa=0}^{k-1} \frac{1}{2^\kappa}. \end{aligned}$$

At time $k_0 = 0$ consider two initial conditions in Cartesian coordinates $\xi_1 = (M, 0)$ and $\xi_2 = (M + \pi, 0)$ for some $M \in \mathbb{R}_{>0}$ as shown in Figure 1. The corresponding polar coordinates for ξ_1 and ξ_2 are $(r_1^0, \theta_1^0) = (M, 0)$ and $(r_2^0, \theta_2^0) = (M + \pi, 0)$. Then, the initial increment is $|\xi_1 - \xi_2| = \pi$. At the time instance $k = 1$, the polar coordinates are $(r_1(1), \theta_1(1)) = (\frac{M}{2}, M)$ and $(r_2(1), \theta_2(1)) = (\frac{M+\pi}{2}, M + \pi)$, hence, the angle difference is π. Consequently, the distance between two states is $|\phi(1; 0, \xi_1) - \phi(1; 0, \xi_2)| = r_1(1) + r_2(1) = \frac{M}{2} + \frac{M+\pi}{2} = M + \frac{\pi}{2}$. Since $M \in \mathbb{R}_{>0}$ is arbitrary, for any function $\beta \in \mathcal{KL}$, we can always choose a sufficiently large M such that (2) is violated. Thus, system (17) is not incrementally stable. □

The above example exploits the fact that global asymptotic convergence of any two trajectories to each other (incremental stability) is a stronger property than global asymptotic convergence of all trajectories to a single trajectory (convergent dynamics). This is a direct consequence of the triangle inequality.

The following theorem provides a connection from convergent dynamics to incremental stability by restricting the state space of system (1) to be a compact and positively invariant set.

Theorem 11. *Suppose system* (1) *is uniformly convergent on a compact and positively invariant set* $\mathbb{G} \subset \mathbb{R}^n$. *Further assume that the right hand side* f *of system* (1) *is locally Lipschitz continuous in* x *on* $\mathbb{G}$. *Then, system* (1) *is uniformly asymptotically incrementally stable on* $\mathbb{G}$.

The proof of Theorem 11 is contained in the appendix.

Conversely, incremental stability does not imply convergent dynamics as shown by the following example adapted from [19, Example 4].

Example 2. The system

$$x(k+1) = -\frac{k}{2} - 1 + \frac{x(k)}{2}, \text{ with } x(k_0) = \xi \in \mathbb{R} \tag{18}$$

has the explicit solution

$$\phi(k; k_0, \xi) = \frac{k_0}{2^{k-k_0}} - k + \frac{\xi}{2^{k-k_0}}, \quad \forall k \geq k_0. \tag{19}$$

For any two initial conditions $\xi_1, \xi_2 \in \mathbb{R}$,

$$\begin{aligned} &|\phi(k; k_0, \xi_1) - \phi(k; k_0, \xi_2)| \\ &= \frac{|\xi_1 - \xi_2|}{2^{k-k_0}} = \beta(|\xi_1 - \xi_2|, k - k_0), \end{aligned}$$

where $\beta \in \mathcal{KL}$ is defined by $\beta(s, r) = \frac{s}{2^r}$ for all $s, r \in \mathbb{R}_{\geq 0}$. Hence, system (18) is uniformly globally asymptotically incrementally stable. However, for the specific initial condition $(\xi, k_0) = (0, 0)$, it is straightforward that the solution (19) from this initial condition is unbounded. Therefore, system (18) is not uniformly globally convergent. □

The above example exploits the fact that convergent dynamics requires the existence of at least one bounded solution. Whereas solutions of an incrementally stable system can be unbounded.

The following theorem provides a connection from incremental stability to convergent dynamics with an assumption that the state space of system (1) is compact and positively invariant.

Theorem 12. *Suppose system* (1) *is uniformly globally asymptotically incrementally stable and there exists a compact and positively invariant set* $\mathbb{G} \subset \mathbb{R}^n$ *under* (1)*. Then system* (1) *is uniformly globally convergent.*

The proof of Theorem 12 is contained in the appendix.

The following theorem provides a sufficient condition for system (1) to satisfy all three convergence properties. It is the discrete-time analogue of the Demidovich result [6, Theorem 1].

Theorem 13. *Assume the right hand side* f *of system* (1) *is continuously differentiable in* x*. Suppose there exists a positive definite matrix* P *such that the matrix*

$$J(k, x) := \frac{\partial f}{\partial x}(k, x)^T P \frac{\partial f}{\partial x}(k, x) - \rho P \tag{20}$$

is negative semidefinite uniformly in $(x, k) \in \mathbb{R}^n \times \mathbb{Z}$ *for some* $\rho \in (0, 1)$*. Then system* (1) *is uniformly globally asymptotically incrementally stable and globally contracting. Furthermore, if there exists* $c \geq 0$ *such that*

$$\sup_{k \in \mathbb{Z}} |f(k, 0)| = c < \infty, \tag{21}$$

then system (1) *is uniformly globally convergent.*

The proof of Theorem 13 is contained in the appendix.

4.2. **Contraction Analysis vs. Incremental Stability.** Assuming the right hand side of system (1) is continuously differentiable, we demonstrate that contraction analysis is a strictly stronger property than asymptotic incremental stability. In fact, contraction analysis is equivalent to exponential incremental stability.

Theorem 14. *Suppose the right hand side of system* (1) *is continuously differentiable in* x *on* $\mathbb{R}^n$*. System* (1) *is uniformly globally exponentially incrementally stable if and only if it is uniformly globally contracting.*

The proof of Theorem 14 is contained in the appendix.

4.3. **Convergent Dynamics vs. Contraction Analysis.** System (17) in Example 1 is globally convergent, however, it is not globally contracting because it is not differentiable at $x = 0$. Hence, a uniformly globally convergent system is not necessarily globally contracting.

However, even in the case where the system dynamics are continuously differentiable, global convergent dynamics is still different from contraction analysis. This is due to the fact that contraction analysis requires an exponential convergence rate whereas convergent dynamics only requires asymptotic convergence. The following example, adapted from [31, Example 1], provides a (time-invariant) system that is asymptotically convergent but not exponentially convergent.

Example 3. The system

$$x(k+1) = f(x(k)) := \frac{x(k)}{\sqrt{x(k)^2+1}}, \text{ with } x(k_0) = \xi \in \mathbb{R} \tag{22}$$

has the solution

$$\phi(k; k_0, \xi) = \frac{\xi}{\sqrt{(k-k_0)\xi^2+1}}, \quad \forall k \in \mathbb{Z}, k \geq k_0. \tag{23}$$

The zero solution $\bar{x}(k) = 0$ is (uniformly) globally asymptotically stable. Indeed, for the Lyapunov function $V(x) = x^2$, we have, for all $x \neq 0$,

$$V(f(x)) - V(x) = -\frac{x^4}{x^2+1} < 0. \tag{24}$$

Hence, system (22) is (uniformly) globally convergent. By inspection of (23), we see that the convergence to the zero solution is not exponential, hence, by Theorem 14, system (22) is not globally contracting. □

Example 4. Returning to Example 2, system (18) is not convergent since the solution passing through $(k_0, \xi) = (0, 0)$ is unbounded. However, system (18) is globally contracting. Indeed, take $\Theta(k, x) = 1$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}$, $\mu = \frac{3}{4}$, and note that $\frac{\partial f}{\partial x}(k, x) = \frac{1}{2}$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. With $F(k, x)$ given by (7) we see that

$$F(k, x)^T F(k, x) - 1 = -\mu.$$

Hence, system (18) is globally contracting. □

Using the results of Theorem 12 and Theorem 14, we provide sufficient conditions under which global contraction implies convergent dynamics.

Theorem 15. *Suppose the mapping f of system* (1) *is continuously differentiable in x on a compact and positively invariant set $\mathbb{G} \subset \mathbb{R}^n$. If system* (1) *is uniformly contracting in $\mathbb{G}$, then system* (1) *is uniformly convergent in $\mathbb{G}$.*

The proof of Theorem 15 is contained in the appendix.

5. Conclusions

This paper contributes discrete-time, time-varying, smooth Lyapunov function characterizations for incremental stability, convergent dynamics, and contraction analysis. In addition, the paper also contributes examples of systems that highlight the essential differences as well as similarities among the three considered notions of stability. Moreover, with appropriate assumptions, we present several conditions that provide connections between each of the three considered convergence properties. Overall, assuming the right hand side of the considered system is continuously differentiable, the relationships between the three convergence properties can be summarized as in Figure 2.

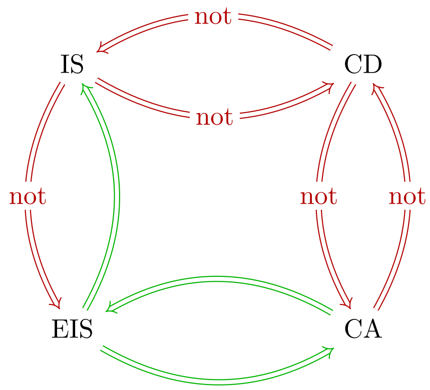

FIGURE 2. Relationships between different convergence properties assuming continuously differential dynamics f. Note the abbreviations: IS for (globally asymptotically) *Incremental Stability*, EIS for (globally) *Exponentially Incremental Stability*, CD for (uniformly globally) *Convergent Dynamics*, and CA for (global) *Contraction Analysis*.

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Appendix

Proof of Theorem 8. Consider the augmented system

$$\begin{cases} x_1(k+1) = f(k, x_1(k)) \\ x_2(k+1) = f(k, x_2(k)) \end{cases} \tag{25}$$

as in [2]. The *diagonal* is the set $\Delta := \{[x^T, x^T]^T \in R^{2n} : x \in \mathbb{R}^n\}$. Let $z = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathbb{R}^{2n}$, $x_1, x_2 \in \mathbb{R}^n$, then the distance from z to the diagonal is given by

$$|z|_\Delta := \inf_{w \in \Delta} |w - z| = \frac{1}{\sqrt{2}} |x_1 - x_2|, \tag{26}$$

where the equality is shown in [2]. Denote $F\Big(k, \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]\Big) := \left[\begin{smallmatrix} f(k,x_1(k)) \\ f(k,x_2(k)) \end{smallmatrix}\right]$ and rewrite (25) as

$$z(k+1) = F(k, z(k)), \tag{27}$$

where $z = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathbb{R}^{2n}$ and $k \in \mathbb{Z}$.

Appealing to the relationship (26), we see that system (1) is uniformly globally asymptotically incrementally stable if and only if the diagonal set Δ is uniform global asymptotic stability with respect to system (27).

Applying [28, Theorem 1 and Lemma 2.8] which state that the closed set Δ is uniformly globally asymptotically stable with respect to system (27) if and only if system (27) admits a smooth, time-varying Lyapunov function $V : \mathbb{Z} \times \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$ for which there exist functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|z|_\Delta) \leq V(k, z) \leq \alpha_2(|z|_\Delta), \tag{28}$$

$$V\Big(k, F(k, z)\Big) - \ V(k, z) \leq -\alpha_3(|z|_\Delta) \tag{29}$$

hold for all $z \in \mathbb{R}^{2n}$ and $k \in \mathbb{Z}$. Using (26), it is clear that (28)–(29) are correspondingly equivalent to (10)–(11).

We conclude that system (1) is uniformly globally asymptotically incrementally stable if and only if system (1) admits a Lyapunov function satisfying (10)–(11). ∎

Proof of Theorem 9. Assume that system (1) is uniformly globally convergent as in Definition 3, so that there exists a solution $\bar{x}(k)$ bounded for all $k \in \mathbb{Z}$; i.e., $\sup_{k\in\mathbb{Z}} |\bar{x}(k)| < \infty$. Applying the converse result of [28, Theorem 1 and Lemma 2.8] with a change of coordinates $z(k) = x(k) - \bar{x}(k)$ to the system

$$\begin{aligned} z(k+1) &= f\big(x(k)\big) - f\big(\bar{x}(k)\big) = f\big(z(k) + \bar{x}(k)\big) - f\big(\bar{x}(k)\big) \\ &=: g\big(k, z(k)\big), \end{aligned}$$

yields a smooth function $W : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|z|) \leq W(k, z) \leq \alpha_2(|z|), \text{ and} \tag{30}$$

$$W\big(k+1, g(k, z)\big) - W(k, z) \leq -\alpha_3(|z|), \tag{31}$$

hold for all $z \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Reverting to the original coordinates and defining $V(k, x) = W\big(k, x - \bar{x}(k)\big)$, we obtain (12)–(13) from (30)–(31). Moreover,

$$0 \leq V(k, 0) \leq \alpha_2(|\bar{x}(k)|) \leq \alpha_2\Big(\sup_{k\in\mathbb{Z}} |\bar{x}(k)|\Big) =: c < \infty$$

establishes (14). This completes the proof of the first statement of the theorem.

Conversely, if (12)–(14) hold, then applying the forward result of [28, Theorem 1], we see that system (1) is uniformly globally asymptotically stable with respect to the trajectory $\bar{x}(k)$. Thus, it is only left to show that $\bar{x}(k)$ is bounded and unique. Indeed,

$$|\bar{x}(k)| \leq \alpha_1^{-1}(V(k, 0)) \leq \alpha_1^{-1}(c) < \infty$$

shows that $\bar{x}(k)$ is bounded. Lastly, uniqueness of $\bar{x}(k)$ follows from Remark 5. ■

Proof of Theorem 10. The proof proceeds by demonstrating the equivalence of an exponential incremental Lyapunov function and exponential incremental stability and then appealing to Theorem 14.

Similar to the proof of Theorem 8, consider the augmented system (25) (also, its shorthand notation (27)). Since (26) holds, system (1) is uniformly globally exponentially incrementally stable if and only if the diagonal set Δ is uniformly globally exponentially stable with respect to system (27).

Applying [28, Theorem 2] which states that a closed set Δ is uniformly globally exponentially stable with respect to system (27) if and only if system (27) admits a continuous, time-varying Lyapunov function $V : \mathbb{Z} \times \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$ for which there exist constants $c_1, c_2, c_3 \in \mathbb{R}_{>0}$ such that

$$c_1|z|_\Delta \leq V(k,z) \leq c_2|z|_\Delta, \tag{32}$$

$$V(k, F(k,z)) - V(k,z) \leq -c_3|z|_\Delta \tag{33}$$

hold for all $z \in \mathbb{R}^{2n}$ and $k \in \mathbb{Z}$. Using (26), it is clear that (32)–(33) are correspondingly equivalent to (15)–(16). Therefore, we conclude that system (1) admits an (exponential incremental) Lyapunov function (15)–(16) if and only if system (1) is uniformly globally exponentially incrementally stable.

Applying Theorem 14 yields that system (1) is uniformly globally exponentially incrementally stable if and only if system (1) is globally contracting. Note that the proof of Theorem 14 does not employ any prior results to Theorem 10 or Theorem 10 itself. This completes the proof. ■

Proof of Theorem 11. The proof follows that of [19, Theorem 8]. Denote the diameter of $\mathbb{G}$ as $d_{\mathbb{G}} := \max_{x,y\in\mathbb{G}} |x-y|$. Fix any $\xi_1, \xi_2 \in \mathbb{G}$. Applying the triangle inequality, we see that

$$\begin{aligned} &|\phi(k;k_0,\xi_1) - \phi(k;k_0,\xi_2)| \\ &\leq |\phi(k;k_0,\xi_1) - \bar{x}(k)| + |\phi(k;k_0,\xi_1) - \bar{x}(k)| \\ &\leq 2\beta(d_{\mathbb{G}}, k-k_0) \end{aligned} \tag{34}$$

for all $x \in \mathbb{G}$ and $k \geq k_0$. Let $L \in \mathbb{R}_{>0}$ be the maximum Lipschitz constant for f on $\mathbb{G}$; i.e.,

$$|f(k_0,\xi_1) - f(k_0,\xi_2)| \leq L|\xi_1 - \xi_2| \tag{35}$$

for all $\xi_1, \xi_2 \in \mathbb{G}$. It immediately follows from (35) that

$$|\phi(k;k_0,\xi_1) - \phi(k;k_0,\xi_2)| \leq L^{k-k_0}|\xi_1 - \xi_2|. \tag{36}$$

Combining (34) and (35), we see that

$$\begin{aligned} &|\phi(k;k_0,\xi_1) - \phi(k;k_0,\xi_2)| \\ &\leq \min\left\{L^{k-k_0}|\xi_1-\xi_2|, 2\beta(d_{\mathbb{G}}, k-k_0)\right\}. \end{aligned}$$

From there, we can obtain a function $\hat{\beta} \in \mathcal{KL}$ such that

$$|\phi(k;k_0,\xi_1) - \phi(k;k_0,\xi_2)| \leq \hat{\beta}(|\xi_1 - \xi_2|, k-k_0)$$

for all $\xi_1, \xi_2 \in \mathbb{G}$ and $k \geq k_0$.

Therefore, system (1) is uniformly asymptotically incrementally stable on $\mathbb{G}$. ■

Proof of Theorem 12. Suppose system (1) is uniformly globally asymptotically incrementally stable, then by applying Theorem 8, there exists a function V satisfying (10)–(11).

To derive the existence of a bounded solution $\bar{x}(k)$, we use a discrete-time version from [13, Lemma 2] of Yakubovich's Lemma [21, Lemma 2] which states if there exists a compact and positively invariant set $\mathbb{G}$ of system (1), then there exists a solution $\bar{x}(k)$ of system (1) defined on $\mathbb{Z}$ and satisfying $\bar{x}(k) \in \mathbb{G}$ for all $k \in \mathbb{Z}$. Thus, we see that

$$\begin{aligned}&\alpha_1(|x-\bar{x}(k)|) \leq V(k,x,\bar{x}(k)) \leq \alpha_2(|x-\bar{x}(k)|),\\ &V\Big(k+1, f(k,x), f(k,\bar{x}(k))\Big) - \ V(k,x,\bar{x}(k))\\ &\quad \leq -\alpha_3(|x-\bar{x}(k)|)\end{aligned}$$

for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Moreover, since $\bar{x}(k)$ is bounded, we have

$$V(k,0,\bar{x}(k)) \leq \alpha_2(|0-\bar{x}(k)|) < \infty \tag{37}$$

for all $k \in \mathbb{Z}$.

Hence, the function $W(k,x) := V(k,x,\bar{x}(k))$ satisfies conditions (12)–(14). Thus, by applying Theorem 9, we conclude that system (1) is uniformly globally convergent. ■

Proof of Theorem 13. Consider any two initial conditions $\xi_1, \xi_2 \in \mathbb{R}^n$. The straight line segment connecting two points ξ_1 and ξ_2 at time k_0 is parametrized by a function $\gamma_{k_0} : [0,1] \to \mathbb{R}^n$

$$\gamma_{k_0}(s) := s\xi_1 + (1-s)\xi_2, \tag{38}$$

where $s \in [0,1]$. Then, at time k_0+1, a parametrized curve initiated from the segment (38) is defined by

$$\begin{aligned}\gamma_{k_0+1}(s) &:= \phi(k_0+1; k_0, \gamma_{k_0}(s))\\ &= \phi(k_0+1, k_0, s\xi_1 + (1-s)\xi_2)\\ &= f(k_0, \gamma_{k_0}(s)).\end{aligned} \tag{39}$$

Since the segment (38) is continuously differentiable with respect to s, and f is continuously differentiable with respect to x, the curve (39) is continuously differentiable with respect to s. Applying the mean value theorem, there exists an $\bar{s} \in [0,1]$ such that

$$\begin{aligned}f(k_0,\xi_1) - f(k_0,\xi_2) &= \gamma_{k_0+1}(1) - \gamma_{k_0+1}(0)\\ &= \frac{d}{ds}\gamma_{k_0+1}(s)\Big|_{s=\bar{s}},\end{aligned} \tag{40}$$

where the first equality follows directly from (39).

Differentiating (38) with respect to s and evaluating at $s = \bar{s}$, we see that

$$\frac{d}{ds}\gamma_{k_0}(s)\Big|_{s=\bar{s}} = \xi_1 - \xi_2. \tag{41}$$

From (40), (39), the chain rule, and (41), we obtain

$$\begin{aligned}
&f(k_0,\xi_1) - f(k_0,\xi_2)\\
&= \frac{d}{ds} f(k_0, \gamma_{k_0}(s))\Big|_{s=\bar{s}}\\
&= \frac{\partial f}{\partial x}(k_0,x)\Big|_{x=\gamma_{k_0}(\bar{s})} \frac{d}{ds}\gamma_{k_0}(s)\Big|_{s=\bar{s}}\\
&= \frac{\partial f}{\partial x}(k_0,x)\Big|_{x=\gamma_{k_0}(\bar{s})} (\xi_1 - \xi_2).
\end{aligned} \tag{42}$$

With the positive definite matrix P, define

$$\begin{aligned}
&Q := Q(\xi_1, \xi_2, k_0, \bar{s}) =\\
&\frac{\partial f}{\partial x}(k_0,x)\Big|^T_{x=\gamma_{k_0}(\bar{s})} P \frac{\partial f}{\partial x}(k_0,x)\Big|_{x=\gamma_k(\bar{s})}.
\end{aligned}$$

Hence, (42) implies

$$\left\| f(k_0,\xi_1) - f(k_0,\xi_2) \right\|_P^2 = \left\| \xi_1 - \xi_2 \right\|_Q^2. \tag{43}$$

From condition (20), we see that

$$\left\| \xi_1 - \xi_2 \right\|_Q^2 \le \rho \left\| \xi_1 - \xi_2 \right\|_P^2$$

which, with (43), implies

$$\left\| f(k_0,\xi_1) - f(k_0,\xi_2) \right\|_P^2 \le \rho \left\| \xi_1 - \xi_2 \right\|_P^2. \tag{44}$$

Define a Lyapunov function candidate $V(k, \xi_1, \xi_2) := \|\xi_1 - \xi_2\|_P^2$. Since P is positive definite, V satisfies condition (10). Next, using (44),

$$\begin{aligned}
&V(k_0+1, f(k_0,\xi_1), f(k_0,\xi_2)) - V(k_0,\xi_1,\xi_2)\\
&\le -(1-\rho)V(k_0,\xi_1,\xi_2),
\end{aligned}$$

for all $\xi_1, \xi_2 \in \mathbb{R}^n$ and $k_0 \in \mathbb{Z}$. This implies that V satisfies condition (11). Therefore, by virtue of Theorem 8, system (1) is uniformly globally asymptotically incrementally stable.

It is straightforward to see that the function V defined above also satisfies (15) and (16) of Theorem 10. Hence, system (1) is globally contracting.

The additional condition (21) together with (44) implies that system (1) is uniformly globally (exponentially) convergent by applying [11, Theorem 1].■

Proof of Theorem 14. "$\Longrightarrow$" Assume system (1) is globally uniformly uniformly exponentially incrementally stable. Let $\begin{bmatrix} \phi(k;k_0,\xi) \\ \phi_\delta(k;k_0,\xi_\delta,\xi) \end{bmatrix}$ denote the semi flow of system (9a)–(9b) for the initial condition $\begin{bmatrix} \xi \\ \xi_\delta \end{bmatrix} \in \mathbb{R}^{2n}$, whereby we note that the solutions to the dynamics (9b) (denoted by ϕ_δ) depend on the dynamics (9a), so that ϕ_δ depends on a third parameter ξ identifying a particular reference trajectory of (9a). Parametrize the straight line segment connecting ξ and $\xi + \xi_\delta$ at time k_0 by a function $\gamma_{k_0} : [0,1] \to \mathbb{R}^n$ given by

$$\gamma_{k_0}(s) := \xi + s\xi_\delta, \tag{45}$$

where $s \in [0,1]$. At time $k \geq k_0$, we denote the parametrized curve initiated from segment (45) by

$$\gamma_k(s) := \phi(k; k_0, \xi + s\xi_\delta) = \phi(k; k_0, \gamma_{k_0}(s)).$$

We make the following claim:

Claim 1. The trajectory $\left[\begin{smallmatrix}\phi(k;k_0,\xi)\\ \phi_\delta(k;k_0,\xi_\delta,\xi)\end{smallmatrix}\right]$ of system (9a)–(9b) is identical to $\left[\begin{smallmatrix}\phi(k;k_0,\xi)\\ \frac{d}{ds}\phi(k;k_0,\gamma_{k_0}(s))\end{smallmatrix}\right]$; i.e., $\phi_\delta(k; k_0, \xi_\delta, \xi) = \frac{d}{ds}\phi(k; k_0, \gamma_{k_0}(s))$ for all $k \geq k_0$.

Proof of the claim. By the chain rule,

$$\frac{d}{ds} f(k, \phi(k; k_0, \gamma_{k_0}(s)) = \frac{\partial f}{\partial x}(k, x) \frac{d}{ds}\phi(k; k_0, \gamma_{k_0}(s)),$$

which implies that $\frac{d}{ds}\phi(k; k_0, \gamma_{k_0}(s))$ satisfies the displacement dynamics (9b). Thus, it is only left to prove that the initial conditions of both trajectories are the same; i.e., $\frac{d}{ds}\phi(k_0; k_0, \gamma_{k_0}(s)) = \xi_\delta$. Note that $\frac{d}{ds}\phi(k_0; k_0, \gamma_{k_0}(s)) = \frac{d}{ds}\gamma_{k_0}(s)$. Hence, by taking the derivative both sides of (45) with respect to s, we obtain the desired result. □

For any sufficiently small $\epsilon > 0$, Definition 2 yields $k \geq 1$, $\lambda > 1$ so that

$$\begin{aligned} &|\phi(k; k_0, \gamma_{k_0}(s + \epsilon)) - \phi(k; k_0, \gamma_{k_0}(s))| \\ &\leq \kappa |\gamma_{k_0}(s + \epsilon) - \gamma_{k_0}(s)| \lambda^{-(k-k_0)}. \end{aligned} \tag{46}$$

Dividing both sides of (46) by ϵ then sending $\epsilon \to 0$, we obtain

$$|\phi_\delta(k; k_0, \xi_\delta, \xi)| = \left|\frac{d}{ds}\phi(k; k_0, \gamma_{k_0}(s))\right| \leq \kappa|\xi_\delta|\lambda^{-(k-k_0)}. \tag{47}$$

Thus, from the global exponential incremental stability of the subsystem (9a), by considering (9a)–(9b), we have established the global exponential stability of the subsystem (9b). As this system is linear, its global exponential stability is the same as global exponential incremental stability.

In what follows, we will construct a nonsingular matrix $\Theta(k, x)$ and, hence construct $F(k, x)$ such that (6) is satisfied for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$.

To this end observe that the transfer matrix $\Phi(k, k_0; \xi)$ of the linear displacement dynamics (9b), which satisfies $\Phi(k, k_0; \xi)\xi_\delta = \phi(k; k_0, \xi_\delta, \xi)$ for all $\xi_\delta \in \mathbb{R}^n$ and all $k \geq k_0$, satisfies the exponential bound

$$\|\Phi(k, k_0; \xi)\| \leq \kappa\lambda^{-(k-k_0)} \tag{48}$$

(with $\kappa \geq 1$, $\lambda > 1$) for $k \geq k_0$ *independently* of the initial condition ξ of the reference trajectory generated by (9a). From here we may follow, mutatis mutandis, the proof of [32, Theorem 23.3] and define an $n \times n$ matrix

$$Q(k, \xi) := \sum_{j=k}^{\infty} \left(\Phi(j, k; \xi)\right)^T \Phi(j, k; \xi)$$

and noting that, due to (48), $\|Q(k, \xi)\| \leq \frac{\kappa^2\lambda^2}{\lambda^2 - 1}$ independently of k and ξ, so $Q(k, \xi)$ is well defined. Following the remainder of said proof, one establishes, mutatis mutandis, that for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$,

$$\eta I \preceq Q(k,x) \preceq \rho I, \tag{49}$$

$$\frac{\partial f}{\partial x}(k,x)^T Q(k+1,x)\frac{\partial f}{\partial x}(k,x) - Q(k,x) \preceq -\nu I \tag{50}$$

where η, ρ, and ν are positive constants.

Applying the Cholesky factorization on the uniformly positive definite matrix $Q(k,x)$, there exists a nonsingular matrix $\Theta(k,x)$ such that, $Q(k,x) = \Theta(k,x)^T\Theta(k,x)$. With this construction of $\Theta(k,x)$, condition (5) is automatically satisfied by appealing to (49). Now, multiplying each side of (50) by $\Theta(k,x)^{-T}$ from the left and $\Theta(k,x)^{-1}$ from the right, we obtain

$$\begin{aligned}
&\Theta(k,x)^{-T}\frac{\partial f}{\partial x}(k,x)^T \\
&\times \Theta(k+1,x)^T\Theta(k+1,x)\frac{\partial f}{\partial x}(k,x)\Theta(k,x)^{-1} \\
&- \Theta(k,x)^{-T}Q(k,x)\Theta(k,x)^{-1} \\
&\preceq -\nu\Theta(k,x)^{-T}\Theta(k,x)^{-1}
\end{aligned}$$

which, with (7), implies

$$F(k,x)^T F(k,x) - I \preceq -\nu\Big(\Theta(k,x)\Theta(k,x)^T\Big)^{-1}. \tag{51}$$

Next, since $Q(k,x) \preceq \rho I$ by (49), for any $s \in \mathbb{R}^n$ we have

$$\begin{aligned}
&(\Theta(k,x)^{-1}s)^T Q(k,x)(\Theta(k,x)^{-1}s) \\
&\leq \rho(\Theta(k,x)^{-1}s)^T(\Theta(k,x)^{-1}s).
\end{aligned}$$

Straightforward manipulations result in

$$\frac{1}{\rho}s^T s \leq s^T(\Theta(k,x)\Theta(k,x)^T)^{-1}s.$$

Thus, we arrive at

$$-\nu(\Theta(k,x)\Theta(k,x)^T)^{-1} \preceq -\frac{\nu}{\rho}I. \tag{52}$$

Combining (51) with (52), we see that

$$F(k,x)^T F(k,x) - I \preceq -\frac{\nu}{\rho}I.$$

Therefore, condition (6) is satisfied for this construction of $F(k,x)$ with $\mu := \frac{\nu}{\rho}$, in other words, system (1) is uniformly globally contracting.

"$\Longleftarrow$" Now, assume that (1) is uniformly globally contracting. Define matrix $Q(k,x) := \Theta(k,x)^T\Theta(k,x)$. By the definition of $\Theta(k,x)$, condition (5) implies $Q(k,x)$ is uniformly positive definite; i.e., there exists $\eta > 0$, $\rho > 0$ so that

$$\eta I \preceq Q(k,x) \preceq \rho I. \tag{53}$$

Multiplying each side of (6) with $\Theta(k,x)^T$ from the right and $\Theta(k,x)$ from the left, then expanding by using (7), we have

$$\begin{aligned}&\Theta(k,x)^T\Big(\Theta(k,x)^{-T}\frac{\partial f}{\partial x}(k,x)^T\Theta(k+1,x)^T\\&\times\Theta(k+1,x)\frac{\partial f}{\partial x}(k,x)\Theta(k,x)^{-1}\Big)\Theta(k,x)\\&-\Theta(k,x)^T\Theta(k,x)\preceq-\mu\Theta(k,x)^T\Theta(k,x).\end{aligned}$$

After straightforward simplifications, we see that

$$\begin{aligned}\frac{\partial f}{\partial x}(k,x)^TQ(k+1,x)\frac{\partial f}{\partial x}(k,x)-Q(k,x)\\\preceq-\mu Q(k,x)\preceq-\frac{\eta}{\mu}I\end{aligned}\tag{54}$$

where the second matrix inequality follows directly from the lower bound of (53). Thus, from (53) and (54), applying [32, Theorem 23.3], we conclude that the linear time-varying subsystem (9b) is uniformly exponentially stable. Furthermore, from the first inequality of (54), we have

$$\frac{\partial f}{\partial x}(k,x)^TQ(k+1,x)\frac{\partial f}{\partial x}(k,x)\preceq\beta Q(k,x)\tag{55}$$

for some $\beta\in(0,1)$.

Pick any $\xi_1,\xi_2\in\mathbb{R}^n$. Consider a straight line segment connecting ξ_1,ξ_2 and parametrized by a function $\gamma_{k_0}:[0,1]\to\mathbb{R}^n$, at k_0, given by

$$\gamma_{k_0}(s):=s\xi_1+(1-s)\xi_2,\tag{56}$$

where $s\in[0,1]$. The length of this segment at $k=k_0$ is $l_0=|\xi_1-\xi_2|$. Then, for any $k>k_0$,

$$\gamma_k(s):=\phi(k;k_0,\gamma_{k_0}(s))\tag{57}$$

is a curve connecting $\phi(k;k_0,\xi_1)$ to $\phi(k;k_0,\xi_2)$ parametrized by $s\in[0,1]$. The length of the curve defined in (57) at time k is given by

$$l(k)=\int_0^1\sqrt{\frac{d}{ds}\gamma_k(s)^TQ(k,\gamma_k(s))\frac{d}{ds}\gamma_k(s)}ds.$$

Applying the chain rule to $\gamma_{k+1}(s)$ for $s\in[0,1]$ yields

$$\begin{aligned}\frac{d}{ds}\gamma_{k+1}(s)&=\frac{d}{ds}\phi(k+1;k_0,\gamma_{k_0}(s))\\&=\frac{d}{ds}f(k,\phi(k;k_0,\gamma_{k_0}(s)))\\&=\frac{\partial f}{\partial x}(k,x)\frac{d}{ds}\phi(k;k_0,\gamma_{k_0}(s))=\frac{\partial f}{\partial x}(k,x)\frac{d}{ds}\gamma_k(s).\end{aligned}$$

We see that

$$\begin{aligned}
&\frac{d}{ds}\gamma_{k+1}(s)^T Q(k+1,\gamma_{k+1}(s))\frac{d}{ds}\gamma_{k+1}(s) \\
&= \left(\frac{\partial f}{\partial x}(k,x)\frac{d}{ds}\gamma_k(s)\right)^T Q(k+1,\gamma_{k+1}(s)) \\
&\quad \times \left(\frac{\partial f}{\partial x}(k,x)\frac{d}{ds}\gamma_k(s)\right) = \frac{d}{ds}\gamma_k(s)^T \times \\
&\left[\frac{\partial f}{\partial x}(k,x)^T Q(k+1,\gamma_{k+1}(s))\frac{\partial f}{\partial x}(k,x)\right]\frac{d}{ds}\gamma_k(s) \\
&\le \frac{d}{ds}\gamma_k(s)^T \beta Q(k,\gamma_k(s))\frac{d}{ds}\gamma_k(s),
\end{aligned}$$

where the final inequality follows from (55). Therefore,

$$\begin{aligned}
&l(k+1) - l(k) \\
&\le \int_0^1 \sqrt{\frac{d}{ds}\gamma_{k+1}(s)^T Q(k+1,\gamma_{k+1}(s))\frac{d}{ds}\gamma_{k+1}(s)}ds \\
&- \int_0^1 \sqrt{\frac{d}{ds}\gamma_k(s)^T Q(k,\gamma_k(s))\frac{d}{ds}\gamma_k(s)}ds \\
&\le (\sqrt{\beta}-1)l(k)
\end{aligned}$$

which, in turn, implies

$$l(k+1) \le \frac{l(k)}{\lambda},$$

where $\lambda = \frac{1}{\sqrt{\beta}} > 1$ since $\beta \in (0,1)$. Consequently, it is straightforward to see that

$$l(k) \le l_0 \lambda^{-(k-k_0)}$$

and, hence,

$$\begin{aligned}
&|\phi(k;k_0,\xi_1) - \phi(k;k_0,\xi_2)| \le l(k) \\
&\le l_0\lambda^{-(k-k_0)} = |\xi_1 - \xi_2|\lambda^{-(k-k_0)}
\end{aligned}$$

for all $\xi_1, \xi_2 \in \mathbb{R}^n$, $k \ge k_0$, and $k, k_0 \in \mathbb{Z}$, we conclude that (1) is globally exponentially incrementally stable. ■

Proof of Theorem 15. The proof of Theorem 15 is a consequence of Theorem 12 and Theorem 14. In particular, by Theorem 14, we see that a globally contracting system is globally exponentially incrementally stable. Then, by Theorem 12, we conclude that a globally contracting system whose state evolves in a compact and positively invariant set, is uniformly globally convergent. ■

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